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Kerala Mathematics
History and Its Possible Transmission to Europe
Edited by
George Gheverghese Joseph

B.R.P.C.

Kerala Mathematics

History and Its Possible Transmission to Europe

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George Gheverghese Joseph
University of Manchester, U.K.

In the history of mathematics, the invention of calculus and 'the passage to infinity' are seen as major benchmarks in the creation of modern or advanced mathematics. A principal focus of this volume is tracing the beginnings of modern mathematics to India, notably Kerala, and how these ideas and techniques may have been transmitted to Europe. If such transmission proves to be significant, it would help to deconstruct the prevailing Eurocentric account of the development of modern mathematics. To that end invitations were extended to a group of distinguished historians, mathematicians, educators, linguists, philosophers and others to make presentations at a two-day Workshop held in Trivandrum. And this is reflected in the breadth and depth of the papers presented in this volume.

George Gheverghese Joseph was born in Kerala and studied at the University of Leicester and then joined the University of Manchester. He has travelled widely holding university appointments in East and Central Africa, Papua New Guinea and New Zealand as well as a Royal Society Visiting Fellowship in India. In recent years he lectured at Hobart, Monash, Perth and Sydney in Australia; at Cornell, Los Angeles, New Mexico, New York, Berkeley, Chicago and Washington in the United States; at York, Laval and Toronto in Canada and many other universities around the world. In December 2005 he organised an International Workshop in Kerala which was the culmination of an AHRB research project on Kerala Mathematics. He has appeared on Radio and Television programmes in India, United States, Australia, South Africa and New Zealand as well as United Kingdom. In October 2000 he was called to the Bar of the Middle Temple, London. At present he holds joint appointments at University of Manchester and Toronto.

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FOREWORD

This volume is the outcome of an International Workshop on 'Medieval Kerala Mathematics: Historical Relevance and Possibilities of Transmission of Europe' jointly organized by the University of Exeter, UK and Kerala Council for Historical Research (KCHR), at Kovalam near Trivandrum, Kerala, in December, 2005. The Workshop was supported through a project of the Arts and Humanities Research Board [AHRB], UK. Conceived as an interdisciplinary workshop, it was attended by twenty five scholars from Canada, Italy, United Kingdom and India, belonging to disciplines like Mathematics, History and Philosophy.

The workshop was an offshoot of the AHRB project under Dr. George Gheverghese Joseph – eminent scholar and the author of the acclaimed work, *Crest of the Peacock.* His project seeks to explore the historical roots of Kerala Mathematics and the possible transmission of certain calculus elements from Kerala to Europe.

It is well known that medieval Kerala society contributed richly to the cultural moorings of modern Kerala. But its aeonian contributions to architecture, mathematics, astronomy and medicine are yet to be fully comprehended. The scholars of medieval Kerala had commented, retold or adapted many of the practical manuals of the major systems of knowledge of classical India, indicating that they were cognizant of the developments in the realms of knowledge elsewhere. For example, Kautilya's *Arthasastra,* one of the ancient treatises on statecraft, had been retold in Malayalam as early as the 12^{th} century AD. Palm leaf records of the era also vouch for the spurt of intellectual activity and creativity during the times. The period also witnessed remarkable strides in the cultural realm. *Koodiyattam* and *Kuttu,* Kerala's early dance forms, were re-fashioned on uniform

scientific and aesthetic principles during this period. Temples became centers of learning as well as platforms for various performing arts.

Historians have put forward arguments concerning the cultural symbiosis achieved by Kerala society during the period. Multiple religious faiths like Jainism, Buddhism, Vedic Brahmanism, Saivism, Vaishnavism, Islam, Christianity and Judaism peacefully co-existed and interacted, making Kerala society the crucible of a composite culture. Unfortunately only limited attempts have been made so far to understand the socio-cultural background or the intellectual history of the region which registered amazing levels of achievements especially in mathematics and astronomy. Similarly, though we are well aware that Europe was keeping its eyes open to the intellectual developments in this part of the world, very little is known about the nature of exchanges and assimilations that took place between India and Europe.

According to the generally held view, the European mathematicians, Newton and Leibniz independently invented and developed the methods of calculus (an essential element of modern mathematics) in the last 17th century. But what is not so well known is that the works of the Kerala mathematicians Madhava, Nilakantha, and othershad discussed important elements of calculus, including numerical integration methods and infinite series derivations, much ahead of their European counterparts, as early as the 14th and 16th centuries.

The Project of Dr. George Gheverghese Joseph has already identified indirect evidence to suggest that some of the mathematical foundations underlying European calculus could have percolated from India through Jesuit scientists. Direct, clinching evidences corroborating this inference and the routes of the probable transmission other than through the Arabs, remain elusive. The project is an attempt to open up enquiries in these directions. If these conjectures with respect to the transmission of mathematical ideas underlying calculus could be proved, it would trigger off new directions in the historical perspective on the development of Early Modern mathematicians even necessitating a re-evaluation of the works of doyens like Newton, Leibniz and Gregory. It would also create ripples

powerful enough to re-draw the picture of the intellectual history of medieval Kerala.

The Workshop altogether had 8 sessions (including one Technical session), generally of 75 minutes duration. Each session had opening remarks by a discussant for about 15 minutes on an already circulated paper, followed by the author talking on his/her paper and commenting on the views of the discussant for about 30 minutes. The final lap of each session would be a general discussion by the participants for another 30 minutes. The Workshop discussed ten papers and the special technical session had papers on the mathematical activities of the Kerala School as well as a Report on the Project. The participants listened to the presentations on wide-ranging archival investigations undertaken in Italy, Spain, Portugal and other European countries in response to the above questions. The papers were on, the role of the Jesuits as scientific intermediaries, the methodological issues in establishing transmissions of scientific ideas across cultures, and a survey of the work on calculus and the infinite series in seventeenth-century Europe. Two papers on the background of the emergence of the Kerala School of mathematics and astronomy from a historical perspective and another one attempting to link the philosophy and values of mathematics and the Kerala School were also presented.

The plenary session, chaired by Professor Romila Thapar, exmined questions on the future directions of the project. Dr. Thapar while endorsing the significance of the project opined that, the historical aspects of Kerala Mathematics in relation to other parts of the sub-continent also should be studied in depth. She observed that exploration of various conduits of transmission like the Indian Ocean and Arab channels, would throw more light to the importance of the contributions of the Kerala School of mathematics. The participants pointed to the need for comprehensive understanding of the socio-cultural dimensions of medieval Kerala Mathematics. A global approach was suggested by some scholars, as it would help place the Asian contribution in the creation of modern mathematics, in its proper perspective and thus rewrite many Eurocentric notions of modernity.

The papers in this volume are revised versions of the papers presented. Two additional papers related to the theme of the Workshop have also been included. This volume is unique in that it features the drift of the discussion and critique that followed the presentations. As in similar edited volumes, the style (including the presence or absence of diacritical marks) varies considerably with each contributor. The indulgence of the reader is requested on some repetitiveness which is inherent in such a project. The original text, in some papers, has been altered for the sake of clarity and consistency and the editor has done commendable work to make this a palatable intellectual experience.

P. J. Cherian
Director, Kerala Council for Historical Research

INTRODUCTION

George Gheverghese Joseph
University of Manchester, UK

According to historical literature, the general methods of the calculus were invented independently by Newton and Leibniz in the late 17^{th} century using the methods of Greek mathematics. However, what is often forgotten, is that important elements of the calculus including numerical integration methods and infinite series derivations for π, sinx, cosx, $\tan^{-1}x$, were already in place in the works of the Kerala mathematician, Madhava and subsequently elaborated by Nilakantha, Jyesthadeva, Sankara Variyar and others between the 14^{th} and 17^{th} centuries

A history of mathematics may be approached in a number of different ways: as a chronological survey; by tracing the development of a particular theme or subject; through exploration of the life and work of individual mathematicians; or by focussing on specific mathematical communities at a particular time and place. The title of the Workshop: "Kerala Mathematics: History and Possibilities of Transmission," (on which this book is based), could include all these different approaches but the emphasis was on two main themes: a survey of Kerala mathematics (which is but a continuation of the history of Indian mathematics) and an examination of the conjectured transmission of certain fundamental elements of the calculus from Kerala to Europe through the Jesuit conduit.

The general context of the research project of which this book was the culmination, may be traced back to my book "The Crest of the Peacock: Non-European Roots of Mathematics"

first published in 1991. On page 20 of that book, I had remarked:

> *"One of the conjectures posed in Chapter 9 is the possibility that mathematics from medieval India, particularly from the southern state of Kerala may have had an impact on European mathematics of the sixteenth and seventeenth centuries."*

I then went on to add that this conjecture remained to be investigated further.

This one-sentence conjecture aroused considerable interest and controversy. I was asked at a meeting soon after the publication of the book whether I was in the business of dethroning Newton and Liebniz! It was also taken up by some who quoted me as suggesting that the Indians invented calculus!

But many asked a more reasonable question: Who were the agents/ or conduits who took Indian knowledge to Europe. Not the Arabs, surely, since the period in question was later than the generally recognised period when there was transmission of mathematical and astronomical ideas from India to the Arab world. Neither could the transmission have had any significant impact on European mathematics after the middle of the seventeenth century since a proto-calculus had already developed in Europe.

In search of conduits for possible transmission, the Jesuits became a possibility. They were in Cochin and Goa from around the middle of the sixteenth century. They had some of the best training in mathematics of that time in Europe. They realized from the time of their foundation as religious order that countries such as India and China could be impressed by scientific and intellectual prowess. They were also in the forefront of developments in educational, theoretical and practical sciences in Europe. They established a continuous presence in Malabar from the time of the arrival of Francis Xavier in Goa in 1540 to 1670; and those who appeared after 1578 seem to have as a secondary objective the gathering of information about India. So that in the talks that I gave before and after the book came out and in the second edition of the 'Crest of the Peacock' (2000), I

specifically linked the transmission conjecture with the Jesuits in Cochin. And hence the emergence of the Project, funded by the Arts and Humanities Research Board (AHRB) of the United Kingdom.

From the very inception of the Project, we were conscious of the fact that it had to be an interdisciplinary project – involving historians, mathematicians, educators, linguists, philosophers and others. And this is reflected in the breadth and depth of the disciplines of the participants of the Workshop held at the end of the Project in December 2005.

Even before the AHRB Project had begun, we who were members of the Aryabhata Group based at the University of Exeter (including Dennis Almeida and I) had identified some indirect evidence that seemed to indicate that the mathematics underlying the Renaissance calculus could have been obtained from India through the efforts of the Jesuits, motivated by Gregorian calendar reform of 1582 and other concerns relating directly or indirectly to astronomy and navigation.[1] A principal focus of our Project was to research the *later* line of communication between India and Europe that followed the *earlier* transmission of Indian science via the Arabs. Evidence for the early transmissions is fairly well-known. Professor Sarma in the paper that he presented at the Workshop and published in this book provides a useful survey of this subject. If our conjectures with respect to the transmission of mathematical ideas underlying calculus were established, it would signal an important change in the historical perspective of the Early Modern mathematics and would necessitate a re-evaluation of the work of major European figures such as Newton, Leibniz and James Gregory.

A principal aim of this Project was to study the mode and effect of the conjectured transmission of Kerala mathematics to Europe with a view to informing the history of mathematics. This was done by examining the following sets of questions:

- What was the extent of the *transmission* of knowledge by the Jesuits from Kerala to Europe in the 16th and 17th centuries?
- Was Kerala mathematics part of this transmission?

- What was the mode of diffusion of this transmitted mathematics in Europe?

The archival investigation undertaken in Europe followed the prescription stated in the original AHRB bid and summarised in the above questions. That is, follow up the conjecture implied by the *indirect* evidence of Jesuit involvement in the transmission of Kerala scientific knowledge and identify evidence of transmission [via any conduit, Jesuit or otherwise] in the correspondence of Renaissance mathematicians involved in the development of the calculus and/or infinite series.

The investigation broadly adopted the scientific methodology of examining the two hypotheses: The Null Hypothesis [NH] that the development of the calculus in Europe was independent of the earlier development in Kerala and the Alternative Hypothesis [AH] that the development of the calculus in Europe was influenced by the transmission of the earlier Kerala mathematics: and deciding on the basis of the evidence gathered whether or not NH or AH was sustainable.

There are two points to bear in mind here. Firstly, the operation of gathering evidence is a continuing process and thus any conclusions drawn at any point in the Project would necessarily be provisional. Secondly, whatever was the conclusion reached in the final analysis would be of significance to the history of mathematics: If AH is supported by the evidence then its significance is evident; if NH is supported then it will be the first major scientific development in the post-Ancient era that has remained localised in its origin locality [and that despite the existence of a direct corridor of communication between Kerala and Renaissance Europe after the appearance of the Portuguese on the Malabar coast].

At the present time the evidence (the details are contained in the relevant papers of this volume) may seem to support NH but there are sufficient reasons to continue the search. The most pressing reason at this point in time would be that the reports that Jesuits like Matteo Ricci and Antonio Rubino had been obliged to send to Peitro Maffei, the historian of Jesuit activities, have still not been located. Until there is evidence that this correspondence has been lost or destroyed, the search ought to

continue. There has been little *direct* evidence of transmission in the correspondence of the Renaissance mathematicians that have been studied thus far [e.g. Gregory, van Roomen, Viete, Clavius, Grienberger], in the correspondence in the Mersenne network [17 volumes of *Correspondance du Marin Mersenne*]. The correspondence of Italian Renaissance mathematicians [e.g. Torricelli, Cavalieri] remains to be studied. Further, although earlier Indian works [e.g. those of Aryabhata, Brahmagupta, etc] have been located in Leiden, the lack of relevant information [such as the date of acquisition, information about who initially studied/translated the material, etc] prevents any meaningful conclusion being drawn for the purpose of this Project.

In the absence of *direct* evidence, some *indirect* tests have been proposed by Van der Waerden ('the hypothesis of a common origin') and Neugebauer to name just two major historians of mathematics. Neugebauer argued that to establish transmission in the absence of direct evidence requires a suitable chronology for testing priority, the existence of a corridor of communication and the identification of methodological similarities and dissonance. A modification of this methodology for establishing transmission involving legal standards of proof has been proposed in this project and discussed in a paper already referred to.[2]

From an early period of their presence, and much more in the case of China, the Jesuits perceived themselves as mediators of European science. And in part this was reflected in their role as intelligence-gatherers to fill some of the critical gaps that had appeared in Renaissance Europe, notably in the fields of navigation and calendar construction. Some of the Jesuits sent on overseas missions were well qualified in sciences, and among those who were in India even for a short period included Matteo Ricci, Johann Schreck and Antino Rubino.

Before and during the early phase of our Project we discovered some *circumstantial* evidence to support our thesis. These are listed in our paper on pages pp. 265-66 of this volume. Knowledge of local languages and inclusion of local sciences in the Jesuit colleges on the Malabar Coast, description of the sciences and the mechanical arts of the Malabar region sent to Rome, enquiries regarding the Indian calendrical science (Ricci),

reports on the errors in European tables based on inferences from local calendrical knowledge (Rubino), transmission of astronomical observations intended for the benefit of Kepler (Schreck), unsuccessful attempts at learning from the Brahmins the methods of predicting the hour and minute of the eclipses of the sun and moon(Rubino) are all listed in the original documents. These are some of the examples of the Jesuit interest in acquiring knowledge of the local sciences

A list of manuscripts searched in obtaining direct evidence took us to a number of archival sources in Europe, including ARSI and Gregorian University in Rome, University of Coimbra Archives, the Ajuda Library in Lisbon, Oriental Manuscript Library in Leiden.

We went one stage further: We looked at the works of Europe-based Jesuit mathematicians of some ability in mathematics who were in touch with the Jesuit missionary scientists mentioned earlier. A close study was made of Clavius and Grienberger who were both in receipt of correspondence from Ricci and Rubino. We also studied the works of Cristova Borri, a unique figure in that he was the only Jesuit scientist that we identified as spending periods in India before returning to Portugal.

We examined the trigonometric work of Grienberger in some detail (and in particular GES600 and GES 674) from his correspondence with Clavius and searched for its antecedents. We found that Clavius was essentially re-writing the earlier work on trigonometric tables constructed by Regiomontanus who adopted Aryabhata's rule in constructing his sine table. This may however, have come through the Arabs.

A puzzling question initially arose as to how Grienberger in constructing his sine table obtained his initial sine value for 1 minute correct to 22 decimal places. He does not explain this and our initial supposition was that this was obtained through the application of the infinite series for the sine. This would have shown a clear indisputable link between European and the Kerala methods since such a method was unknown in Europe and would take a Newton to discover it. However, with further thought and work, it was found that it is not at all practicable to construct trigonometric tables graduated in degrees using the

infinite series for sine. This proved to be a cautionary lesson for us.

In any case, our work on Clavius's and Grienberger's attempts at constructing trigonometric tables could be of interest to historians of mathematics and work is proceeding at present to complete and publish our results. There were a number of uncatalogued Malayalam and Tamil palm leaf manuscripts in Rome at the Vatican and National Libraries, which were examined by Dr Raghava Variyar. His report on them is presented as a paper in this volume. They were not, on the whole, of any mathematical or astronomical interest.

Our tentative conclusion after the painstaking trawl of these manuscripts was that so far we have found no *direct* evidence exists for the conjectured transmission. This still remains a provisional conclusion since not all the materials required to be studied have as yet been studied. As Professor Baldini indicates in his paper in this volume there may be materials in private libraries and uncatalogued documents in public libraries which have not been examined. There is also the possibility of oral transmission, as indicated by Dr Balasubramanian in his paper in this volume, of computational rules of practitioners of navigation as a mode of transmission between Kerala and Europe. We need to look at this possibility more carefully in the future.

We should also look into the context and the motives of the Jesuit missionaries who were sent to India and their mode of communication with one another. The primary motive was of course evangelical, but to achieve these different strategies were adopted. As Professor Baldini indicated in his paper, there was a difference in the allocation of the Society's scientific experts between India and China on the basis that the sociological contexts of the Indian and Chinese science were different. The emphasis in India on the part of the Jesuits was on a high level of teaching and debating philosophy and theology. The activities and accomplishments of the Jesuit De Nobili in Tamil Nadu bear ample testimony to this viewpoint.[3] In the case of China it was felt that there was a need to impress the Chinese court with their knowledge of astronomy and mathematics. Having said that, there was also the intention to undertake scientific intelligence gathering missions as illustrated by the examples of Ricci and

Rubino, to name just two notable Jesuits in Cochin, who are mentioned in our paper and that of Professor Baldini.

It is worth again emphasizing the point that if the null hypothesis (NH) of no *direct* transmission is sustained by further evidence gathered that may be gathered in the future, it will be the first major case of a scientific development in more recent times that has remained localized in its place of origin despite the existence of a corridor of communication, presence of strong motivation to learn from indigenous sources and the existence of opportunities to do so given the linguistic facility of the Jesuits and their privileged position in the Court of Cochin.

Moving next to other types of evidence of transmission, an important part of the *indirect* test of transmission involves a comparative analysis of the methodology used by both the Kerala and renaissance European mathematicians in deriving their calculus and infinite series. Further, evidence had to be sought from the existence of Kerala mathematics framed within the European epistemology to establish the mode and chronology of the conjectured transmission.

The first of the substantive topics to be investigated as part of the comparative analyses of the two mathematical traditions was the construction of sine tables. This was chosen because we found in the 16th century Europe, a group of mathematicians, notably Rheticus, van Roomen and Christopher Grienberger at the Collegio Romano, were engaged in efforts to construct increasingly accurate tables, based on an unexplained highly accurate seed value for sine 1' correct to 20 decimal places. One of the papers presented by Dr Mallayya and myself on pages 104-108 of this volume is based on of a detailed examination of the history of constructing sine tables in India[4], starting with Aryabhata's *Aryabhatiya* (AD 499) and finishing with Nilakaṇtha's *Golasara* (c. AD 1500), .which involved English translations and interpretation of key sections of various texts. Sine tables of length 24, 48, 96, 192, 384,... were generated using Nilakantha's method and then compared with the corresponding modern values to obtain an accuracy of 18 to 20 places of decimals in the case of many of the values. The question that arose was whether the seed value of the sine or the

methodology used in European tables bore any similarities to those derived from the Kerala tables.

The next substantive topic to be investigated in Kerala mathematics was the work on infinite series relating to circular functions (or the π series) The project was fortunate in being able to make use of Mr Rajasekhar and Dr Mallayya, both scholars of old Malayalam and Sanskrit. With their linguistic skills and mathematical background, they were able to translate certain key sections from the seminal Kerala text, the *Yuktibhasa*, and explain the Kerala derivations of the arc tan series and its special case the series for π. This facilitated a comparative study of the similarities or otherwise in the derivation of the series in both Kerala and Indian mathematics. The studies of both scholars are presented as separate papers in this volume. Dr Mallayya also carried out an extensive study of the 'tools' used by the Kerala mathematicians in their derivation of infinite series for circular and trigonometric functions, notably the 'rule of three', the right-angled triangle theorem, the summation of a geometric series and establishing the behaviour of a quotient taken to its limit. Only a brief indication of the work on the last two topics is given in the paper shown on pages 90-94 of this volume.

A final substantive topic to be investigated was the social and historical context in which the Kerala School of mathematics and astronomy developed between the 14th and 16th centuries. For this purpose an extensive trawl was made of both Malayalam and other published and archival sources on the history of medieval Kerala. In particular, special attention was paid to the intellectual history and institutions of the period. To help us in the task, we were fortunate to find Dr Vijalakshmy, a local historian with the requisite linguistic skills to examine archival material relating to Jesuits and other agents. Dr Vijaylakshmy's paper is included among the Workshop papers; and so is another paper by Dr Mallayya on the background of the members of the Kerala School of Mathematics and Astronomy. Personal details were sought regarding notable members such as Madhava, Paramesvara, Nilakantha, Sankara Variyar, Jyesthadeva and Achuta Pisharoti. Their social background was investigated with certain specific issues addressed:

(i) Were the Nambuthri Brahmins who were members of the School generally younger sons in their families and therefore belonged to a leisurely class in general?

(ii) How did non-Brahmins such as Sankara Variyar and Achuta Pisharoti become such notable members of the Kerala School?

(iii) How important was caste in determining the membership of the School? Were there contacts between the traditional astrologers and those who belonged to the School?

(iv) Did the fact that the founder of the School, Madhava, belonged to the Empran caste have any relevance for the constitution and dynamism of the School?

(v) Was the fact that the *Yuktibhasa* (the seminal text of the Kerala School) was composed in Malayalam of any special significance? Why was Malayalam a preferred language of communication and dissemination in this context?

(vi) What was the status and relationship between pursuit of astronomy and other forms of scholarship of that time?

(vii) What part did the Temple play in disseminating the knowledge of the Kerala School? Did the individual members of the Kerala School have much contact with their local Temple?

(viii) What was the nature of the relationship between the Kerala astronomers and others in South India? Was the relationship exemplified by that between Sundararaja and Nilakantha as indicated in the book *Sundararaja-prasnottara* (which was in the form of questions in astronomy posed by the former and answered by the latter)?

Another important issue relates to possible contacts between the Jesuits and mathematicians and astronomers of Kerala. On this the evidence is somewhat sparse and anecdotal and may be summarized as:

- The Nambuthiris served as intermediaries between the Portuguese and the Cochin court.
- There is some documentary evidence of Jesuit scientists actively seeking the assistance of Brahmins in obtaining knowledge of various Indian sciences.
- There is evidence of the Jesuits actively engaged in evangelical activities among the Brahmins and in the Cochin Court and a record of even recruiting a Brahmin into their Order.
- There is evidence that that some Jesuit scientists in Cochin learnt to read and write Malayalam and Sanskrit at an institution set up in Cochin for this purpose.
- Finally, there is evidence that the Jesuits had close contacts with influential personnel of the various Royal Court.

We have sought but not found any contemporary documents regarding the Jesuits activities in the Court of Cochin, including possibly foraging for indigenous knowledge of astronomy and mathematics. This would require further primary sources research in the archives of Kerala.

THE STRUCTURE AND THE SUMMARY OF THIS VOLUME

The Workshop on which this volume is based covered five related themes offering thirteen papers in total. The names of the presenters and the titles and summaries of the papers are given below. Most papers conclude with a short summary of the discussion that followed after the papers were presented

1. Kerala School of Mathematics and Astronomy: The Social and Historical Contexts

- George Gheverghese Joseph "Infinite Series across Three Cultures: Background and Motivation - A Brief Survey"

  (This paper contains a short survey of the social and cross-cultural contexts in which the work on infinite series arose in Kerala between the 14^{th} and 16^{th} centuries

by comparing this work with those in China during the eighteenth century and the European work beginning in the 17th century.)

- V. Madhukar Mallayya and George Gheverghese Joseph "Indian Mathematical Tradition: The Kerala Dimension"

  (This paper discusses Kerala mathematics from a historical point of view. Building on earlier work in the North, Kerala mathematics and astronomy attained a new height through the work of an array of accomplished mathematicians such as Madhava, Paramesvara, Nilakantha Somayaji, Jyesthadeva, Sankara Variyar and others. Their singular contribution in mathematics was in their work on infinite series relating to circular and trigonometric functions.)

- Vijayalekshmy. M and George Gheverghese Joseph "Intellectual Background of Medieval Kerala With Special Reference To Mathematics And Astronomy"

  (This paper deals with the cultural and intellectual aspects of Kerala's historical development. It highlights the contribution of the Kerala Brahmins in a number of fields of intellectual endeavours. Outside influences also enriched the cultural tapestry: many Arab influences were introduced in the practical field of knowledge like the *rahmania* tradition of the Arab trade percolated into the society. The paper also notes the *guru-sishya parampara* (i.e., the teacher – student relationship) of the Kerala School and its role in sustaining a long tradition of scientific learning.)

2. **The Work of the Kerala School of Mathematics and Astronomy: Some Exemplars**

- Jean Michel Delire "The Mathematics Of The *Tantrasangraha* And Its Successors"

- V.Madhukar Mallayya and George Gheverghese Joseph "Kerala Mathematics: Motivation, Rationale and Method"
- P. Rajasekhar "Derivation Of The Infinite Series Expansion For π as Demonstrated In The *Yuktibhasa*

  (At this technical session, three papers were considered, with the copies of the first two made available before the Workshop. The major presentation was made by Dr Mallayya who dealt with three topics: (a) construction of sine tables based on the methods of Nilakantha as contained in his *Golasara* manuscript; (b) the derivation of the infinite series relating to arc tan (and its special case the π series); and (c) the notion of proof in Kerala mathematics. He was at pains to point out that contrary to popular views regarding Indian mathematics, Kerala mathematicians adopted rigorous methods of proof to validate their results. They often employed algebraic rationale, geometric demonstration and numerical verification to justify the outcome of their intellectual enquiry, as illustrated in the papers of Dr Delire and Mr Rajasekhar. All three papers involved the identification, study and translation of original manuscripts.)

3. **Establishing Transmissions and Other Philosophical and Methodological Issues**

- Arun Balasubramaniam "Establishing Transmissions: Some Methodological Issues"

  (This paper proposes that if direct evidences of transmissions are lacking there could have been some other channels through which the knowledge was transmitted to Europe. One possibility may well have been that a set of computing rules transmitted by Indian craftsmen to their European counterparts, rather than theoretical ideas exchanged between Indian and European intellectuals. Hence, Indian mathematical discoveries may have reached Europe through a set of

practical computing rules via Indian cartographers, navigators and calendar makers to their European counterparts.)

- Paul Ernest "The Philosophy Of Mathematics, Values, and Keralese Mathematics"

  (The paper begins by emphasizing the point that in the western world, epistemological value of mathematics was considered supreme. It then proceeds to argue that the narrow intellectual focus of the philosophy of mathematics on epistemology or Platonic ontology to the exclusion of history and practices of mathematics is a somewhat misguided. As a result, over the years, the value of calculations and computations has been undermined. Indeed, mathematical proof- the most important axiomatic feature of mathematics – and calculation techniques are very close. As a concluding note, the paper puts forward the case against Eurocentrism in the history of mathematics and points out that one of the major casualties in the Eurocentric view of mathematics has been the marginalizing or undervaluing the contributions to mathematics of Indian sub-continent, and in particular Kerala.)

4. Earlier Transmission of Mathematics from India and Parallel Developments Elsewhere

- Sreeramula Rajeswara Sarma "Early Transmissions Of Indian Mathematics"

  (In many Arab texts, the mention of Indian decimal system as well as the knowledge of Indian astronomical tables is found, indicating direct evidence of transmission. The paper also makes a case-study of westward transmission of a special mathematical knowledge called the Rule of Three or *Trairasika*. This is also apparent in the case of perpetual motion machine, the paper opines. The paper concludes by emphasizing

the need to study in greater detail transmission of mathematical knowledge within the Indian sub-continent.)

- Jacqueline Stedall "Calculus And Infinite Series In Seventeenth-Century Europe"

  (This paper discusses the major influences that shaped the course of modern mathematics in 17^{th} century Europe, namely the rediscovery and evaluation of the Greek mathematical texts and the Islamic influences, particularly emanating from the Moorish Spain. These two influences were synthesized by Francoise Viete and his ideas were later developed by others such as Fermat, Roberval and Decartes. So that around 1600, all the key elements were in place for Newton and Leibniz to take the stage and for Europe to take a big leap towards infinite series and modern calculus. Both these major figures in different ways contributed to the discoveries in calculus and infinite series and the routes that they took were also very different. In conclusion, the paper raises an important point that is particularly relevant to the present project. Since the European approach was so different from that of the Indian both in calculus and infinite series, what would be useful would be to look at the two as an interesting case of parallel developments independent of each other.)

5. The Jesuit Conduit: Background and Evidence

- Dennis Almeida and George Gheverghese Joseph "A Summary Report On The Investigation On The Possibility Of The Transmission Of The Medieval Kerala Mathematics To Europe"

  (This paper reports on the search for evidence regarding the possibility of the transmission of the medieval Kerala mathematics to Europe. Despite the paucity of direct documentary evidence, there is mounting circumstantial

evidence to support-to-support transmission. The paper draws attention to the facts that Jesuit was interested and sought knowledge relating to the computational, astronomical and calendrical practices of the local people. In their attempt to obtain such local knowledge and promote their evangelical activities, the Jesuits learned the local language and gained influence in the Cochin Royal Court. To determine whether there were any direct evidences a meticulous search was undertaken in the archives of Rome. There was also an investigation to unearth possible relevant material from the correspondences of Jesuits such as Ricci and Rubino. Additionally, an initial survey was done of the catalogued and uncatalogued palm leaf manuscripts in vernacular languages such as Malayalam kept in the Vatican and other libraries in Rome as well as in other places in Europe. The paper concludes with the observation that if the evidence gathered does not support the transmission hypothesis, the Kerala phenomenon may well be the first major case of scientific development in post ancient era that remained localized in its place of origin despite the existence of a direct corridor of communication to Europe.)

- Ugo Baldini "The Jesuit Mathematicians In India (1578-1650) As Possible Intermediaries between European And Indian Mathematical Traditions"

  (This paper points out that with the exceptions of individuals such as Matteo Ricci, Antonio Rubino and Johann Schreck, very few Jesuit missionaries who came to India in the period 1578-1650 had adequate mathematical knowledge to comprehend the advanced mathematics of the Kerala mathematicians. Further, teaching and learning higher mathematics was never on the agenda of the Jesuits colleges in India and also there is no evidence of Jesuit missionaries knowing Sanskrit prior to 1649. In its concluding section, the paper makes an important point Existing evidence point to the fact that the more scientifically inclined missionaries were sent to China as the Jesuits were while in India, the emphasis was on theological studies.)

- M.R. Raghava Variyar "Uncatalogued Malayalam Manuscripts In Europe: A Report Of Work Carried Out In Two Libraries In Rome"

  (This paper consists of a report by Dr Variyar regarding his search in Bibliotec Apostolic Vaticana (BAV) and Biblioitc Nazinale Centrale di Roma (BNCR) and also in the Bibliotheque Nationale de France in Paris. An extensive survey of palm leaf manuscripts written in Malayalam, many of them uncatalogued or in some cases wrongly catalogued, indicate that these manuscripts were of varied nature like dictionaries, grammatical texts, historical writings, and theological writings. Only a very small part of the manuscripts related to scientific, technological and computational aspects. The paper concluded that within India, there are a large number of manuscripts on higher learning including mathematics, which needs to be explored)

ENDNOTES

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1 These are contained in various publications, notably The Aryabhata Group (2002) "Transmission of the Calculus from Kerala to Europe" in *Proceedings of the International Seminar and Colloquium on 1500 Years of Aryabhateeyam,* Kerala Sastra Sahitya Parishad, Kochi: pp. 33-48 and D.F. Almeida and G.G. Joseph (2004) "Eurocentrism in the History of Mathematics: the Case of the Kerala School", *Race and Class*, 45: 45-59.

2 See The Aryabhata Group (2002), op. cit pp. 33-48

3 For further details on the activities of Roberto de' Nobili Teotonio, see R. de Souza & C. J. Borges (eds.) *Jesuits in India: A historical perspective*, Macau 1992, p. 212-213).

4 As part of this Project, Dr Mallayya undertook a substantive study of the construction of sine tables from Aryabhata to Nilakantha. This is now in the process of being prepared for publication.

PART – I

The Kerala School of Mathematics and Astronomy: The Social and Historical Contexts

1

INFINITE SERIES ACROSS CULTURES: BACKGROUND AND MOTIVATION - A Brief Survey

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Two powerful tools contributed to the creation of modern mathematics in the seventeenth century: the discovery of the general algorithms of calculus and the development and application of infinite series techniques. Many introduced to the general algorithms of calculus; consisting of differentiation, integration and other related techniques, know that the names normally associated with the development of that stream are Newton and Leibniz. The other stream was the discovery and applications of infinite series and again the European names such as Mercator, Wallis, Gregory, Newton and Leibniz predominate. These two streams of discovery reinforced each other in their simultaneous development because each served to extend the range of application of the other.

However, what is less known is that the origin of the analysis and derivations of certain infinite series, notably those relating to the arctangent, sine and cosine, was not in Europe, but an area in South India which now falls within the state of Kerala. From a region covering less than a thousand square kilometres north of Cochin and during the period between the fourteenth

and sixteenth century, there emerged discoveries in infinite series, which predate similar work of Gregory, Newton and Liebniz by at least 200 years.

There are a number of questions worth asking about the activities of this group of mathematicians/astronomers (who will be hereafter referred to as the Kerala School) apart from technical ones relating to the mathematical content of their work, which will be dealt by papers in Section II of this volume. In this paper, we will consider specific questions relating to the social landscape in which the Kerala School developed, the mathematical motivation underlying their interest in a particular series, the arc tan series (and its special case, the π series). In conclusion, the paper attempts to provide a cross-cultural context, by comparing the Kerala work with those in China during the eighteenth century and only briefly mention the European work of the seventh century since it has had adequate exposure in literature.

The direct inspiration for Kerala mathematics were the works of Aryabhata and his commentators, notably Bhaskara I. In AD 499 at the age of 23, Aryabhata, composed his seminal text *Aryabhatiya*. An Arabic translation of the text entitled *Zij al-Arjabhar* was made around AD 800. The influence of the astronomical and mathematical ideas in the text, both inside and outside India cannot be overestimated. His influence was strongly felt in Kerala for about a thousand years. When we talk about the Kerala work we have in mind the period from about the birth of Madhava (c. AD 1350) to about 1600 after which there are texts but they are not important.

The story of the discovery of Kerala mathematics sheds some fascinating light on the character of the historical scholarship of the period. In 1832, Charles Whish read a paper to a joint meeting of the Madras Literary Society and the Royal Asiatic Society in which he referred to five works of the period, 1450-1850: The *Tantrasamgraha* ("A Digest of Scientific Knowledge") of Nilakantha (1444-1545), the *Yuktibhasa* ["An Exposition of the Rationale"] of Jyesthadeva (fl. 1500-1610), *Kriyakramakari* ("Operational Techniques") of Sankara Variyar (c. 1500-1560) and Narayana (c. 1500-1575), *Karanapaddati* ("A Manual of Performances in the Right Sequence") of Putumana Somayajin (c. 1660-1740) and *Sadratnamala* ("A Garland of Bright Gems") of Sankara Varman (1800-38).

An important feature of the last four texts is their claim to have derived their main ideas from Madhava (c. 1340-1425) and Nilakantha who are referred to as *acharyas* (or teachers). These authors form part of a tradition of continuing scholarship in Kerala over a period four hundred years from the birth of Madhava in 1340 to the probable death of Putumana Somayajin in 1740. In the present state of knowledge of source materials it is difficult to assign many of the developments to any particular person. The results should be seen as produced by members of the Kerala School as it were, spread over several generations.

To obtain a flavour of their work, consider the following quotation from Jyesthadeva's *Yuktibhasa* relating to the arc tan series. Note that capital Sine (Cosine) are sometimes called Indian sine (cosine) and is the product of radius and our sine (cosine)

> "The product of given Sine and the radius divided by the Cosine is the first result. From the first (and then the second, third, ... etc) results, obtain (successively) a sequence of results by taking the square of the Sine as the multiplier and the square of the Cosine as the divisor. Divide (the above results) in order by the odd numbers 1,3,5,... etc to get the (full sequence of) terms. From the sum of the odd terms, subtract the sum of the even terms. (The result) becomes the arc. In this connection, it is laid down that the (Sine) of the arc of its complement, whichever is smaller, should be taken here (as the 'given Sine'); otherwise, the terms, obtained by the (above) repeated process will not tend to a vanishing magnitude."

Translated into modern mathematical notation, this reads:

$$\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \ldots\ldots\ldots\ldots$$

There are a couple of interesting features about this passage from the *Yuktibhasa*. First, its most reliable version is in Malayalam (the local language of Kerala) and not in Sanskrit. Second, the series whose construction is explained by the quotation is now known as the Gregory series named after James Gregory, a Scottish mathematician, who studied the series in 1671. It may be argued that it should more accurately be called the Madhava-Gregory series, since it has been attributed by a number of members of the Kerala School to their founder and easily precedes the work of Gregory by almost three hundred years.

There is another aspect of the *Yuktibhasa*, which is interesting. As the very name *Yuktibhasa* implies, unlike any other Indian mathematical text known of that and earlier periods, the text contains a detailed exposition of the rationale (or proofs) usually in a verbal form, consisting of a mixture of technical terms and *katapayadi* notation, a refinement of Aryabhata's alphabet-numeral system of notation.

It is from this text that one can put together the derivation of the arc tan series according to Kerala mathematicians. A detailed derivation will not be given here but will be discussed in a later paper. The approach involves the direct rectification of an arc of a circle, i.e., the summation of very small arc segments and reducing the resulting sum to an integral. Together with the Madhava-Gregory series, the Kerala School was able to derive the series for π, known in mathematical literature as the Leibniz series. In all fairness, it should be renamed Madhava-Leibniz series.

It may be worth examining in greater detail the method of direct rectification, which formed the basis of the Kerala approach to the derivation of a number of other infinite series. An interesting geometric technique different from the 'method of exhaustion' used in Greek, Arab and later European mathematics, the method involves sub-dividing an arc into *unequal* parts. In the case of the 'method of exhaustion' there occurs a sub-division of the arc into *equal* parts. A different technique was used in Kerala not because the method of exhaustion was unknown to the Indians. Indeed, it is possible that Aryabhata used the method of exhaustion to arrive at his

accurate estimate of the circumference for a given diameter. The 'exhaustion' method was probably avoided because the calculation involving working out the square roots of numbers at each stage of the calculation was seen as a tedious and time-consuming task. What we have here is an interesting foundational difference between the Indian approach and the Greek-Arab-European approach. In the Indian case, numbers were treated merely as entities to be operated on. The stress was on operations rather than the numbers themselves. As a result Indian mathematics steered clear of any philosophical difficulties with incommensurability. For example, surds (*karani*) were accepted as "proper" numbers probably as early as the time of the *Sulbasutras* (c. 800 – 500 BC) and rules for handling them were then developed. In place of "rational-irrational" classification, a notion of "exact-inexact" numbers seemed to have prevailed.

What motivated the Kerala School to undertake the work that they did? The motivation may be found in a verse from *Aryabhatiya* explains how from a given diameter one calculates the circumference of that circle:

> "Add 4 to 100, multiply by 8, and add 62,000. The result is approximately (?) the circumference of a circle whose diameter is 20,000."

Some Western historians of mathematics have argued that the Indians were not aware of the fact that π could never be exactly determined. This confusion may have arisen because of the early mistranslation of the word *asana* as "approximate" or "rough inaccurate value" as in the quotation given above. The word *asana* is more subtle term than that. What it conveys is the notion of 'unattainability'. Anything "unattainable" can never be reached. Unless one understands this distinction, it is not possible to explain the perennial interest of the Kerala School in arriving at increasingly accurate value of what we now describe as π.

Consider the following passage from Nilakantha's commentary, *Aryabhatiyabhasya* at the beginning of the sixteenth century. It is worth quoting in full.

"Why is only the approximate value (of circumference) given here? Let me explain. Because the real value cannot be obtained. If the diameter can be measured without a remainder, the circumference measured by the same unit (of measurement) will leave a remainder. Similarly, the unit which, measures the circumference without a remainder will leave a remainder when used for measuring the diameter. Hence, the two measured by the same unit will never be without a remainder. Though we try very hard we can reduce the remainder to a small quantity but never achieve the state of 'remainderlessness'. This is the problem."

What the passage indicates is that Nilakantha understood the 'irrational' nature of π. So the question that arose is how would one improve the accuracy of the estimate. The following passage from Sankara and Narayana's *Kriyakramakari* suggests a strategy:

> "Thus even by computing the results progressively, it is impossible theoretically to come to a final value. So, one has to stop computation at that stage of accuracy that one wants and take the final result arrived at by ignoring the previous results."

Indian mathematicians were not generally preoccupied with the philosophical implications of numbers as mathematical objects. Faced with irrationality, they tried to arrive at as accurate an estimate as possible. And this is what was being suggested by the authors of *Kriyakramakari.*

However, in applying an infinite series approach to estimate the circumference, the Kerala mathematicians came across a serious difficulty. The problem is that the Madhava-Leibniz series converges far too slowly. For example, summing the first 19 terms on the right hand side of:

$\pi/4 = 1 - 1/3 + 1/5 - .....$

only gives the inaccurate estimate π as 3.194.

The problem was tackled in two directions: (a) rational approximations by applying corrections to partial sums of the series; and (b) obtaining more rapidly converging series by transforming the original series. There was considerable work in both directions as shown in detail in *Yuktibhasa* and *Kriyakramakari*. To illustrate approach (a) from the *Yuktibhasa,* consider the incorporation of the following correction term to the Madhava-Leibniz series:

$$F(n) = \frac{(n^2+1)}{(4n^3+5)}$$

where n is the number of terms on the right hand side

When this correction is applied for n =11, the implicit estimate of π is 3.1415926529 which is correct to 8 places. Interest in improving the accuracy of the estimate continued for a long time, so that as late as the nineteenth century the author of *Sadratnamala* estimated the circumference of a circle of diameter 10^{18} as: 314,159,265,358,979,324 correct to 17 places!

What this work exhibits is a measure of understanding of the concept of convergence, of the notion of rapidity of convergence and an awareness that convergence can be speeded up by transformations. Similar work entered modern mathematics only as late the end of the 18th century. Other achievements of the Kerala School in mathematics and astronomy involved using similar methods to derive the Sine series, Cosine series and even the ubiquitous Taylor series about two hundred and fifty years before it entered modern mathematics.

To understand the context in which the mathematics and astronomy developed in Kerala, there is need to take a broad look at the social landscape of medieval Kerala society and seek answers to the following questions:

* What was the nature of the social structure of medieval Kerala?
* What was the pivotal role of the Kerala temple?
* How was scientific knowledge acquired and disseminated in medieval Kerala?

Each of these questions could well provide enough subject matter for another paper. These questions are discussed in detail in other papers in this Section of this volume. Briefly, the members of the Kerala School were predominantly Nambuthri Brahmins with a few who came from *ambalavasi* castes, such as the Variyars and the Pisharotis, traditionally associated with specific duties in the temple. Within a mainly two-tier caste system, consisting of Brahmins and Nairs, two institutions operated to strengthen and sustain the economic and social dominance of the Nambuthris to a degree not known elsewhere in India: the *janmi* system of serfdom and land-holding with the Nambuthri control of vast tracts of land owned by temples.

There were other factors that helped to strengthen the economic and social powers of the Nambuthris. The Nairs practised the *marumakkattayam* (matrilineal) system of descent without the formal institution of marriage. Sexual alliances between Nair women and Nambuthiri men were permitted, indeed sometimes encouraged, with children of such unions remaining the sole responsibility of their mother's family. At the same time, the Nambuthris operated a system of patrilineal descent (*makkatayam*), with a form of primogeniture that allowed only the eldest son to inherit land and property and to marry Nambuthri women. The eldest son was also required to provide for the material needs of his siblings consisting of younger brothers and unmarried sisters (of whom there were a number given the operation of the system).

Madhava and all those who knew and followed him lived and worked in large compounds called *illams* in villages with predominantly Nambuthri settlements. Set well away from roads to prevent contact with others, often surrounded by a high wall, each *illam* had its own well for water, a tank for bathing and a number of outbuildings. Many of these *illams* belonged to households that owned large landed properties and were very affluent. With their estates farmed by workers or tenants from lower castes and often under the management of Nairs, the Nambuthris, and particularly the younger sons, enjoyed considerable leisure and were expected to pass their time in study and ritual observances.

These *illams* provided a base for the education of the young in Sanskrit works, including mathematical and astronomical classics, notably the *Aryabhitiya* of Aryabhata (b. AD 476) and its commentaries. Not only was traditional knowledge transmitted in these *illams* by rote, but they also provided a centre for research and scholarship. Sometimes, the scholars wrote commentaries on the classics and in those commentaries they appended their own discoveries as additions and supplements. The short distances between the *illams*, the role of the temple and political stability combined to provide for long and stable development, usually based on generations of teacher-student relationships. A study of their interaction with certain temple personnel (especially, the *ambalavasis* such as Sankara Variyar and Achuta Pisharoti) may shed light both on how non-Brahmin Hindus were recruited into their circle as well the process by which a wider dissemination of the results of their work in mathematics and astronomy took place into the neighbouring areas, notably today's Tamilnadu.

Now even a cursory examination of the social background of the members of the Kerala School would indicate that many were Nambuthri Brahmins. But Madhava was not one. He was an Empran Brahmin: a member of a group who were trying very hard to be identified as Nambuthris. Yet he was pursuing activities such as studying mathematics and astronomy, which *per se* did not constitute "high" status activities. The most notable member after Madhava, Nilakantha, belonged to the highest rank among the Nambuthris. He was a *somayaji*, one of the select sub-caste among the Nambuthris, who could carry out the *soma* sacrifices. But there were also other members of the Kerala School who were not Brahmins, the most notable of them being Sankara Variyar and Achuta Pisharoti. This would indicate that the Kerala School consisted of an unusual group interested in mathematics and astronomy that did not have great social value or status, a group that to some extent cut across caste lines and a group who had considerable interactions with the temple personnel. The temple may have fulfilled an important purpose: it served as an institution for acquiring and disseminating knowledge. It was an influential organisation since it combined religious power with secular power, being in many cases

powerful landlords in their own right. The temple served as a medium through which the Nambuthris exerted their power and kept other groups in check.

Another aspect, delving into the background of the members of the Kerala School, would indicate that a number of the Nambuthris might have been younger sons. This fact could lead to an interesting theory. As mentioned earlier, the Nambuthris followed a strict system of primogeniture, so that all landed property went to the eldest son. There was the additional twist. Only the eldest son could marry a Nambuthri woman. The younger sons never married but formed sexual partnerships with Nair women. So we had a situation of a number of Nambuthris freed of all economic and family responsibilities, a truly leisured class.

From these facts it would be a simple matter to posit the following scenario. A group of younger sons, who had very little to do and coming from extremely well-off circumstances, especially since in the Kerala context then, the Nambuthris were also the biggest landlords exerting their control directly or through the temple, were able to live a life of leisure. They had no family responsibilities and their religious duties were confined to a few and not very demanding rituals. Some wrote erotic poetry and a many others whiled away their time in other pursuits. But there were a few who pursued their interest in astronomy and mathematics over a period of about three hundred years, sustained by the institution of the *Gurukula* (a Guru-based and directed system of education) prevalent among the Nambuthris and others of that time. While this explanation is attractive, it would seem somewhat simplistic. First, it does not account the presence and the important influence of non-Nambuthris as members of the Kerala School. Second, this explanation ignores the symbiotic nature of the relationship between the traditional *jyotish* (astrologer) who often came from the lowly Kaniyan caste and the Nambuthri *jyotish*. Third, the *granthaveri* (i.e., written records) of Kerala of this period is full of information about the metrical precision of a number of artisans and craftsmen (such as the carpenter, the trader, the builder and the architect). A number of them showed some awareness of the developments taking place in astronomy and

mathematics during that period. The *granthaveri* records remain a good but relatively untapped source of information about the 'calculating people' of the period.

Incidentally, a study of the social context of Kerala mathematics has an additional bonus. There is a deeply entrenched notion in standard histories of mathematics that all non-European mathematics is utilitarian. A number of Indian scholars have fallen into the same trap. They search for the motivation behind Kerala mathematics in astronomy, navigation and other practical pursuits. One should never ignore the practical motivation. After all many have the members of the Kerala School were both mathematicians and astronomers. The texts of that period cover both subjects. However, a lot of the work on infinite series do not have any direct applications to astronomy. So what led them on in their pursuit of knowledge? I have this vision, of a group of 'pure' mathematicians sitting around in Kerala between the 14th and 16th centuries, like Hardy and Littlewood at the University of Cambridge in the early decades of the twentieth century, indulging in their passion and probably boasting of the fact that the mathematics that they did was of no use to anyone! Some of them probably delighted in long and tedious calculations, such as the one reportedly undertaken by Madhava in calculating the Sine tables to 12 places of decimals! About a hundred years later the Arab mathematician, Jamshid al-Kashi, working in the Samarkand Observatory obtained an implicit estimate for pi, correct to 16 places of decimals, by circumscribing a circle by a polygon having 805 306 368 sides! There seemed also in this case to a veritable fascination with numbers and a boundless delight in calculation which was far removed from any utilitarian concern.

The mathematics produced by the Kerala School was not trivial nor elementary in any sense. And this would bring into question another stereotype regarding Indian mathematics. Standard histories of mathematics would want us to believe that mathematics in India which was elementary and involved mainly arithmetic, virtually came to a stop with Bhaskara II in the 12th century. The existence and content of Kerala mathematics would question this interpretation.

An important reason for taking a cross-cultural perspective in examining the development of a particular area in mathematics is that it provides a useful indication of differences in methods and motivations between different mathematical traditions. In Europe, the details of the circumstances and ideas leading to the discovery of the arctan series by Gregory and Leibniz are well-known. It was an important event because it was a precursor of calculus. In an attempt to discover an infinite series representation of any given trigonometric function and the relationship between the function and its successive derivatives, Gregory stumbled on the arctan series. He took, in terms of modern mathematical notation,

$$d\theta = \frac{d(tan\theta)}{(1 + tan^2\theta)}$$

and carried out term by term integration to obtain his result. Leibniz's discovery arose from his application of fresh thinking to an old problem, namely quadrature or the process of determining a square that has an area equal to the area enclosed by a circle. In applying a transformation formula (similar to the present-day rule for integration by parts) to the quadrature of the circle, he discovered the series for arc tan. It must be pointed out, however, that the ideas of calculus such as integration by parts, change of variables and higher derivatives were not completely understood then. They were often dressed up in geometric language with, for example, Leibniz talking about "characteristic triangles" and "transmutation".

The Chinese work is interesting for a different reason. Infinite series, as a mathematical object, was introduced into China divorced from its European context, i.e., calculus. The introduction of European mathematics into China began in the closing decades of the sixteenth century, when the Chinese first came in contact with the Jesuits. In their intention to spread their religion in China, the Jesuits arrived from Europe bringing with them both new technological gadgets as well as scientific theories which, though not updated with more recent discoveries in Europe, proved a sufficient novelty and attraction for the educated classes. In 1601, the Italian Jesuit, Matteo Ricci (1552-

1601) began his translation of the first six books of Euclid's *Elements* into Chinese in 1607. Later, in the last few decades of the Ming dynasty, many astronomical books were translated into Chinese. But most of the scientific books translated were pre-Newtonian publications. In early Qing dynasty, after listening to a debate between a Jesuit astronomer, Adam Schall, and a Chinese astronomer, Yang Guangxian, the Kangxi Emperor, became interested in Western science. In answer to an invitation to send more mathematicians and astronomers, Louis XIV of France sent a group led by J. de Fontaney, "the King's mathematician", and asked them to make astronomical observations, study the flora and fauna, and learn the technical arts of China. In 1690, the French Jesuits began teaching mathematics to the Emperor and his courtiers. Pierre Jartoux, a French Jesuit, arrived in China in 1701 and taught at the court. He introduced three results new to Chinese mathematics: the power series for sine, versed sine and for pi, which was derived from arc sine function. The last result is attributed to Newton. For none of these results did Jartoux provide a proof; the calculus needed was not known in China until the middle of the nineteenth century.

Ming Antu (d. 1765) was an astronomer who had worked with the Jesuits in cartography and later on reforming the astronomical system. At his death he was the director of the Imperial Board of Astronomy. In his book, *Ge Yuan Mi Lu Jie Fa* (Quick Methods of Trigonometry and for Determining the Precise Ratio of the Circle) contains the statement and proof of nine formulae, including the "three formulae of Master Jartoux". It is possible that Ming Antu was introduced to the three formulae by Jartoux himself. His proofs are based on the generalisation of a method occurring both in Chinese and European tradition: the method of the division of the circle. In China, it is found in Liu Hui's commentary on the premier text, *Chiu Chang Suan Shu,* from the beginning of the Christian Era. The idea of the method is to approximate the circle by inscribing polygons, the number of sides which is doubled at each step. This method was extended by using continued proportions (*lu*) as an algebraic language, so that it applies to the measurement of any arc. In 1720, Takebe Katahiro, a Japanese mathematician,

expressed for the first time the square of the length of the arc of the circle as an infinite power series of the *sagitta* (or the cosine of the half angle). Both the Chinese and Japanese derivation was heavily based on their common mathematical tradition.

In conclusion, there is an intriguing connection between Kerala mathematics and the early work of Srinivas Ramanujan. If one examines some of the early works of Ramanujan (i.e., before he went to England), these are a number of problems published in a Indian mathematics journal that concerned the π series. Some of this work reminds us of the Kerala work. This is a conjecture worth pursuing further. Some rationale for this connection does exist. Ramanujan came from the Iyengar Brahmin caste. The Iyengars are found right across what we would today describe as North Kerala (although it was not part of Kerala but of the Madras Presidency at that time) and in the adjoining Tamil Nadu. Ramanujan's mother was, according to contemporary accounts, a well-known local *jyotish*. She practised her arts not only in individual homes but in local temples as well. A *jyotish* is usually well versed in calculation techniques. So the conjecture is rather than treat Ramanujan as a freak, consider his background, including the possibility that he may have been doing 'ethno-mathematics' which combined h his natural ability with what he learnt from the two English texts to form the basis of his remarkable work later. Now there are reported cases of Iyengars and Nambuthris particularly around the areas of northern Kerala, mixing together within the temple. The temple was, as mentioned earlier, an important centre for dissemination of knowledge.

We can extend our speculation further: Kerala mathematics travelling West. This is the subject of other papers contained in Section V of this volume.

2

INDIAN MATHEMATICAL TRADITION – The Kerala Dimension

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INTRODUCTION

It is generally acknowledged that the 'classical' period of Indian mathematics and astronomy began with Āryabhaṭa I (b. 476). Through his two influential treatises the *Āryabhaṭīya* and the *Āryabhaṭasiddhānta* (the latter is now known only through citations in later works), *Āryabhaṭa* began the task of systematization and synthesis of the mathematical and astronomical knowledge available at that time.[1] In the *Āryabhaṭīya*, the author has included a separate section entitled *Gaṇitapāda* to enunciate the mathematical prerequisites for carrying out astronomical studies. Later authors followed this practice of incorporating mathematics into their standard treatises on astronomy. And these included, between the sixth and twelfth centuries, scholars such as Varāhamihira, Bhāskara I, Brahmagupta, Lalla, Govindasvāmin, Skandasena, Bhaṭṭotpala, Pṛtuḍhakasvāmin, Vaṭeśvara, Śaṅkaranārāyaṇa, Mahāvīra,

Śrīdhara, Āryabhaṭa II, Śrīpati, Udayadivākara, Sūryadeva, and Bhāskara II.

After Bhāskara II (b. 1114), there was supposedly a hiatus in astronomical and mathematical activity in India, although individuals continued to produce influential works. These included Thakkura Pheru (AD 1265 – 1330) of the court of the Delhi Sultanate who wrote the *Gaṇitasāra*; Nārāyaṇa Paṇḍita who composed an important mathematical treatise entitled the *Gaṇitakaumudi* in AD 1356; Mahendra Suri who composed the *Yantrarāja* in AD 1370 which gives a table of ninety Rsines with R = 3600; Jñānarāja who wrote the *Siddhāntasundara* in AD 1503; Nityānanda who wrote the *Siddhāntarāja* in AD 1639; Munīśvara who authored the *Siddhānta-sarvabhauma* in AD 1646; Kamalākara who wrote the *Siddhānta Tatva Viveka* in AD 1658; and Jagannātha Sāmrāt who wrote the *Rekhāgaṇita* in AD 1718 (a Sanskrit translation of the Arabic version of Euclid's *Elements*) and the *Siddhāntasāmrāt* in AD 1732 (a translation of Ptolemy's *Almagest*).

However, during this period, astronomy and mathematics attained new heights in Kerala. Among the major mathematicians/astronomers of the Kerala School, to be discussed in greater detail in later sections of this paper, were Saṅgamagrāma Mādhava, Vaṭaśreṇi Parameśvara, Gārgya Kerala Nīlakantha Somayāji, Jyeṣthadeva, Tṛkkuṭṭaveli Śaṅkara Vāriyar, Putumana Somayāji, and the last in this illustrious line, Śaṅkara Varman. Henceforth, they will be referred to by their underlined names.

In the early nineteenth century, Charles M. Whish of the East India Company came across four manuscripts, namely the *Tantrasaṅgraha* of Nīlakaṇṭha, *Yuktibhāṣā* of Jyeṣṭhadeva, *Karaṇapaddhati* of Putumana Somayāji and the *Sadratnamāla* of Śaṅkara Varma. The information obtained by Whish from these manuscripts was presented at a meeting of the Royal Asiatic Society of Great Britain and Ireland held on 15th December 1832 and published in 1835 as part of the Transactions of that Society. His revelations throw light on the quadrature of a circle using several infinite series methods. However, even earlier, in 1825, Colonel J. Warren's *Saṅkalita* drew attention to Benjamin Heyne's references in 1805 to the same work in Kerala. These

revelations of Warren and Whish remained unnoticed (or hidden) for more than a century until contemporary scholars managed to excavate this knowledge.[2]

The most comprehensive account of the Kerala literature on astronomy, astrology and mathematics available at present is found in K.V. Sarma's "*History of the Kerala School of Hindu Astronomy*". Some discussion of the historical background of the astronomy and mathematics is also found in works on history of literature such as *Kerala Sāhitya Caritram* by Ulloor S. Parameśvara Iyer, *Keralīya Samskṛta Sāhitya Caritram and Kerala Sāhitya Caritram: Carcayum Pūrṇavum* by Vaṭakumkūr Rāja Rāja Varma, *Malayala Bhāṣā Caritram* by P. Govinda Pillai and *Kerala Bhāṣā Sāhitya Caritram* by Nārāyaṇa Panickar.

THE ORIGINS OF THE KERALA SCHOOL

In AD 683 (*Kali* 3785), an assembly of astronomers was called at Tirunāvāya on the banks of river Bhāratapuzha in Northern Kerala on the occasion of the 12-yearly twenty eight day festival called *Māmāṅka mahotsavam*, to discuss the shortcomings of the prevailing astronomical methods of Āryabhaṭa and his School and to reform the system by simplifying the Āryabhaṭan methods.[3] It is believed that at this meeting, some of the methods of the earlier legendary astrologer and astronomer Vararuci I (4th Century AD) were formally adopted by those present. As the author of 248 *Candravākyas* ("lunar sentences") or *Vararuci vākyas*, Vararuci had earlier popularized their use to describe, through a series of mnemonic words or phrases, the positions of the moon at regular intervals each day to help worshippers carry out their daily observances and rituals.[4] Vararuci was the promulgator of the famous *Kaṭapayādi* system of decimal numeration which is an improvement on the older Bhūtasaṅkhyā system.[5] Since it easily applies to Malayalam alphabets it became very much popular in south India.

A prominent participant at the meeting of astronomers at Tirunāvāya was Haridatta (fl. 650 -700) who introduced a new astronomical computation system called the *Parahitha* system through his well-known works *Grahacāranibandhana* and *Mahāmārganibandhana*. Although the Āryabhaṭan system had

been initially adhered to in calculating the planetary positions, it was soon recognized that there was significant variation between the computed and observed values of longitudes of the planets. And hence Haridatta's *Parahitha* system of computation was adopted. It remained in popular use for fixing auspicious times for rituals and ceremonies even after the introduction of the more accurate *Dṛggaṇita* system by *Parameśvara* in the fifteenth century.[6]

Haridatta also Introduced other improvements to the *Āryabhaṭan* system, like popularizing the use of the more versatile *Kaṭapayādi* system of notation in preference to the *Bhūtasaṅkhyā* system or the more complicated alphabetical system of notation of Āryabhaṭa.[7] He devised a system of constructing graded tables of the sines of arcs of anomaly (*manda-jyā*) and of conjugation (*śīghra-jyā*) at intervals of 3°45' to facilitate the computation of the true positions of the planets.

There were several other scholars, following in the footsteps of Haridatta, who worked in the disciplines of astrology, astronomy and mathematics who will not be discussed here. By the beginning of the ninth century, Kerala had undergone some important political changes. The Kulaśekharas in the person of Kulaśekhara Ālvar had re-established their rule in Kerala. The history of this dynasty came to light only recently as a result of the study of the inscriptions of that age.

The burgeoning interest in astronomy was given a further boost during the reign of the second in the Kulaśekhara dynasty, Sthānu Ravi Varman (844 - 885). He established an observatory containing a giant armillary sphere in the capital of the Empire, Mahodayapuram (the present-day Kodungallur), under the charge of Śaṅkaranārāyaṇa (c. 825 – 900), the court astronomer. Śaṅkaranārāyaṇa was a student of Govindasvāmin (c.800 – 850) and like his teacher applied the Āryabhaṭan planetary model in his calculation, basing it on Bhāskara I's *Laghubhāskarīya*. The ruler, Ravi Varman, a keen student of astronomy, asked certain insightful questions which were answered by Śaṅkaranārāyaṇa in his influential commentary on the *Laghubhāskarīya*.

Govindasvāmin, who was the court astronomer of the King Ravi Varma, prepared a detailed *bhāṣya* (or commentary)[8] on the

Mahābhāskarīya, an authoritative work of Bhāskara I (c. 600) on the Āryabhaṭa's system of astronomy. This text contains some interesting mathematics, including rules for second-order interpolation to estimate intermediate sine values for different intervals, which turn out to be a special case of the general Newton-Gauss interpolation formula.[9]

For about three hundred years after *Govindasvāmin* no major figure seemed to appear in Kerala. Indeed, hardly any records exist for sketching developments in mathematics and astronomy from the eleventh to thirteenth. However, it was a period of great activity in other parts of India where mathematician-astronomers, such as Śrīdhara (fl. 900), Āryabhaṭa II (fl. 950), Śrīpati (fl. 1040) and the famous Bhāskara II (b. 1114), were in the forefront. Some of their work must have slowly percolated into Kerala.

The next major figure was Talakkulam Govinda Bhaṭṭatiri[10], a progenitor of the famous Pāzhūr Kaṇiyār family of astrologers, who lived between 1237 and 1295. Born in the village of Ālathūr (three kilometers south of Tirur), one of the thirty-two Brahmin settlements in Kerala established during the seventh and eighth centuries, there is a legend that after performing a *bhajan* (a special act of worship) to his deity in the Trissur temple, Govinda was given the gift of foretelling the future. A *swamiyar* (a high priest) who consulted Govinda was told that he had to contend with three additional births, incurred as a result of arousing the anger of Lord Krishna, being in order of their occurrence a rat-snake, a bull and a tulsi plant. He is said to have left Kerala for Paradeśam (possibly in present-day Tamil Nadu) and studied under a scholar by name of Kānchanūr *Ā*zhvar.

Govinda's major work was *Daśādhyāyī*, a commentary on the first ten chapters of an astrological text *Horā* by the astronomer Varāhamihira (c. 505-587) and this is generally recognized as the most important of the seventy other known commentaries on it. A lesser known work, *Muhūrtaratna*, was referred to by *Parameśvara*, an important figure in the Kerala School, who described the author as a notable astrologer of his time. Indeed, a famous family of astrologers in Kerala, the *Kaṇiyāns* of Pāzhūr, claims their astrological prowess to their supposed descent from Govinda.[11] Information on the scientific activities during the thirteenth century is scarce. We know that

a Nampūtiri Brahmin by the name of Sūryadeva Yajvan (c. 1191–1250) wrote short commentaries on the works of Āryabhaṭa, Śrīpati, Varāhamihira and on the *Mahābhāskarīyabhāṣyā* of Govindasvāmin, which would suggest that interest in astronomy continued over the period. However, we had to wait for the dawn of the new century to witness the birth of the Kerala School of Mathematics and Astronomy.[12]

THE KERALA SCHOOL: LIVES AND WORKS OF ITS MEMBERS

The fourteenth century saw the emergence of the founder of the Kerala School, Saṅgamagrāma Mādhava (c.1340 - 1425). Little personal details are known about him. He belonged to a priestly class called *Emprāntiri*[13] a sub caste of Kerala Brahmins. However, some information regarding the names of his house and village are provided in his own composition, the *Veṇvāroha*, with its commentary by Acyuta Piṣāraṭi (b. 1550), and in the *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha (b.1443). According to this information Mādhava belonged to the house '*Bakulādhiṣṭitavihāram*' (in Sanskrit) or '*Ilaññininnapalli*' (in Malayalam) in the village Saṅgamagrāma in central Kerala (Sarma, 1956; verse 13). The name of this village Saṅgamagrāma is derived from the name of the Lord Saṅgameśvara whose temple is situated in this particular village. This place is now known by the name Iriññjālakuda (Raja, 1954-6; Ulloor II, 1970, p110) which is near Trichur (or Tṛśśivaperūr) and the Ilaññininnapalli of Mādhava is identified as one of the two present Nampūtiri houses namely Iriññāḍapalli and Iriññāḍavalli (Ulloor II, 1970, p111) about eight kilometers from Iriññjālakuḍa near Kallettumkara railway station.

Mādhava's only surviving works are in astronomy and are mainly concerned with refining the *vākya* system of Vararuci, who was mentioned earlier. In both the *Veṇvāroha* and the *Sphuṭacandrāpti*, Mādhava carried out a revision of the *Candravākyas* of Varurici, calculating the exact positions of the moon, correct to the second, for every 36 minutes of the day. Before Mādhava's work, Vararuci's *Candravākyas* only gave values correct to the minute. Using the cyclic nature of the lunar

vākyas in which nine anomalistic months equal 248 days, Mādhava estimated the lunar longitude at nine equally distant times in one day. He also discussed the computation of the longitudes of the planets and of the ascendant.

Besides these works[14], it is from a number of stray verses of Mādhava quoted by those who came after him, that we know of Mādhava's remarkable contribution to the development of Kerala mathematics and astronomy.[15] His fame rests on his discovery of the infinite series for circular and trigonometric functions, notably the Gregory series for arctangent, the Leibniz series for π, and the Newton power series expansions for sine and cosine correct to 1/3600 of a degree. There are also some remarkable approximations, based on the incorporation of "correction terms" to these slowly converging series that have also been attributed to Mādhava.

Mādhava's distinguished pupil was Vaṭaśśeri Parameśvara. He was born in Ālathūr village in 1360. He was a Nampūtiri Brahmin of the Vaṭaśśeri family that specialized in the study of astronomy and astrology. Ālathūr is situated in the Ponnāni taluk of southern Malabar in Malappuram district. The location of this village is mentioned in his work, the *Goḷadīpikā*[16] (II.iv.91). Information about his parentage is lacking but from his commentary on the *Muhūrtaratna* and also from his astrological work *Ācārasaṅgraha* we get information about his grandfather, a well-known astronomer of his time who was a student of the previously mentioned Govinda Bhaṭṭatiri, the author of *Muhūrtaratna* (Sarma, 1963). His son Vaṭaśreṇi Dāmodara was an influential teacher who counted among his students Nīlakaṇṭha and Jyesṭadeva, two major figures of the Kerala School.

Parameśvara wrote a number of commentaries, including ones on Āryabhaṭa's *Āryabhaṭīya*, on Bhāskara I's *Mahābhāskarīya* and *Laghubhāskarīya*, on the *Sūryasiddhānta*, on Govindasvāmin's commentary on the *Mahābhāskarīya*, on Munjāla's *Laghumānasa* and Bhāskarācārya's *Līlāvatī*. Parameśvara's role in scrutinizing and then disseminating the contents of these major texts of Indian astronomy and mathematics cannot be overestimated. As a result, after

Parameśvara, those who followed him had this corpus of knowledge readily available.

Parameśvara's main importance in the development of planetary astronomy in South India is his *Dṛggaṇita* system, explained in his text of that name written in AD 1431. This astronomical work, composed after 55 years of practical observations, computations, corrections and verifications, consists of two parts. The first part deals with the derivations of the mean positions and equations of the centre of the planets, the corrections made and the method for calculating the arc from the sine. The second part merely summarizes the first part using the *Kaṭapayādi* notation. He also wrote *Goḷadīpikā* dealing with various aspects of spherical astronomy of which there are two different versions. The two versions together contain a detailed discussion of the great gnomon, shadow and parallax; the construction of an armillary sphere, the apparent and true motions of planets, methods of measuring the circumference of the earth and other topics on spherics. Between AD 1393 and 1432, Parameśvara made a series of observations of eclipses of the sun and the moon and recorded them in three short works, namely the *Grahaṇamaṇḍana*, *Grahaṇanyāyadīpika* and *Grahaṇāṣṭakạ*. He also wrote on astrology, including a commentary on Govinda Bhaṭṭatiri's *Muhūrtaratna*.

The foundation laid by Parameśvara heralded the emergence of the major figure of Nīlakaṇṭha Somayāji. He was born in AD 1444 into a Nampūtiri Brahmin family of *Comātiris* (i.e., performers of the soma sacrifice) in Tṛkkaṇṭiyūr (*Srīkuṇḍapuram* or *Srīkuṇḍagrāma* in Sanskrit) in the Ponnani Taluk of the Malappuram district as son of Jātavedas (K.S. Shastry, 1930; p180). His family house namely the Keḷallūr has been traced to the present Eṭamana house occupied by distant relations after the extinction of Nīlakaṇṭha's family (Vadakkumkur, 1938; p384). He was a worshipper of Lord Śiva's deity in Śvetāraṇya or Śrīparakroḍa at Tṛpparaṇṇoḍ (as stated in the colophon to *Āryabhaṭīyabhāṣya*). His younger brother Śaṅkara was also well versed in astronomy. Nīlakaṇṭha refers to his brother in some of his works. In commenting on verse 26 in the *Gaṇitapāda* of the *Āryabhaṭīya*, Nīlakaṇṭha refers to his brother's teachings at the house of his patron Netranārāyaṇa (K.S. Shastry, 1930; p156).

Again, at the end of his commentary on the *Goḷapāda*, Nīlakaṇṭha states that he is entrusting the commentary to *Śaṅkara* for its proper propagation (S.K. Pillai, 1957; p156). For his considerable knowledge of astronomy, Nīlakaṇṭha was held in high esteem by his patron and religious head Kauśitaki Netranārāyaṇa Āzhvāñceri Tamprākkal. He married Ārya and had two sons Rāma and Dakṣiṇāmūrti. Dakṣiṇāmūrti was well versed in the *Manusmṛti* and other *Dharmaśāstras* and had mastered three languages - Sanskrit and the two *Dravidian* languages (Tamil and Malayalam). Rāma was the author of *Śrī Rāmāyaṇam* (*Laghurāmāyaṇam*).[17]

In the introductory verse of the *Siddhāntadarpaṇa* and in the commentary on verse 18 the author Nīlakaṇṭha gives the names of scholars from whom he had acquired his knowledge. He learnt *Vedānta Śāstras* from Ravi; the twin sciences astronomy and mathematics from Dāmodara and also from his *Paramguru* (the grand teacher) Parameśvara. The great Malayalam scholar Tuñcattu Ezhuttacchan is believed to have been taught by Nīlakaṇṭha. Among his other illustrious students were two notable members of the Kerala School of mathematics and astronomy, Śaṅkara Vāriyar and Jyeṣṭhadeva. Nīlakaṇṭha wrote a number of influential astronomical works as well as a number of commentaries. His *Goḷasāra* is a short introduction to astronomy, dealing with basic astronomical constants and concepts, the position and movement of planets, computation of sine among other topics. His other writings include the *Siddhāntadarpaṇa* and a detailed commentary on it; an elaborate commentary *Āryabhaṭīyabhāṣya* on the *Āryabhaṭīya* of Āryabhaṭa; the *Candracchāyāgaṇita* and a commentary on it dealing with computations relating to shadow such as computation of shadow length from time (*kramacchāyā*) and time from shadow length (*viparītacchāyā*); the *Tantrasaṅgraha* which is his magnum opus in eight chapters containing several innovative astronomical and mathematical ideas; the *Grahaṇanirṇaya* dealing with computations relating to solar and lunar eclipses; the *Sundararājapraśnottara* which is an astronomical manual giving detailed answers (*uttaram*) or clarifications to certain astronomical questions or problems (*praśna*) raised and addressed to him by a contemporary Tamil

astronomer Sundararāja who had written a computation manual *Vākyakaraṇa*; and the *Jyotirmīmāmsa*, a work containing *Nīlakaṇṭha*'s discussion of astronomical theories.

Nīlakaṇṭha's popularity and contacts with contemporary astronomers outside Kerala is evident from his *Sundararājapraśnottara* (which gives insightful answers to the questions raised by *Sundararāja*). Sundararāja's respectful references to Nīlakaṇṭha as *Ṣaḍdarśani Pāramgatā* (one who has learnt the six systems of philosophy) and *Goḷacūḍamaṇi* etc. in his *Vākyakaraṇa* (Sarma and T.S.K Sastry, 1964; p119) is matched by Nīlakaṇṭha's complimentary references to Sundararāja's commentary on the *Vākyakaraṇa* in his *Āryabhaṭīyabhāṣya* (S.K.Pillai, 1957; p149). This dialogue is important because it provides rare evidence of interest in Kerala mathematics and astronomy in other areas of South India.

Nīlakaṇṭha's fame, however, rests on his work *Tantrasaṅgraha*. This work is a compendium of all the results known up to his time and it generated among those who came after him a number of commentaries, both in Malayalam and Sanskrit, which form an important basis for assessing of Kerala mathematics and astronomy. The work (Sarma, 1977), in eight chapters, containing 432 verses, deals with various topics connected with astronomical calculations and follows the *Dṛggaṇita* system introduced by Parameśvara. It also deals with a variety of subjects including the fixing of the gnomon, calculations of the meridian, of latitude, of the declensions etc., and the prediction of eclipses. To illustrate its range and originality, consider briefly two innovations of Nīlakaṇṭha that have implications for history of astronomy.[18]

In the *Tantrasaṅgraha*, Nīlakaṇṭha carried out a major revision to the *Āryabhaṭan* model for the interior planets, Mercury and Venus. In doing so, Nīlakaṇṭha arrived at a more accurate specification of the equation of the centre for these planets than any other that existed in Arab or European astronomy before Kepler, who was born about one hundred thirty years after Nīlakaṇṭha. In his *Āryabhaṭīyabhāṣya*, Nīlakaṇṭha developed a computational scheme for planetary motion which is better than that of Tycho Brahe in that it correctly takes account of the equation of centre and latitudinal

motion of the interior planets. This computational scheme implies a heliocentric model of planetary motion where the five planets (Mercury, Venus, Mars, Jupiter and Saturn) move in eccentric orbits around the mean Sun which in turn goes round the earth. This model is found to be similar to the one suggested by Tycho Brahe when he revised the heliocentric model of Copernicus. It is significant that all the astronomers of the Kerala School who followed Nīlakaṇṭha, including Jyeṣṭhadeva, Acyuta Piṣaraṭi and Putumana Somayāji accepted Nīlakaṇṭha's planetary model. The other works which he wrote late in life were either commentaries on his earlier works, such as those on his *Candravākya gaṇita* and *Siddhāntadarpaṇa* where the latter consists of a short work in 32 verses dealing with important astronomical constants, the theory of epicycles and other matters of topical interest.

It is likely that Nīlakaṇṭha lived to be over a hundred[19] and during his long life he taught many students, some of them who were to become important figures in their own right. One of his students was Citrabhānu (c.1475 - 1550). He was a Nampūtiri Brahmin from the village of Covvaram (Śivapuram or Śukapuram in Sanskrit, near present-day Trichur). His work, *Karaṇāmṛta* containing four chapters of advanced astronomical calculations within the framework of the *Dṛggaṇita* system, was composed in AD 1530. This work also provides the basics for the preparation of Kerala calendar (*pañcāṅga*) He was also the author of *Ekavimśatipraśnottara* ('Twenty-one Questions and Answers') solving each of a set of 21 pairs of simultaneous equations in two unknowns. The twenty-one pairs arise by taking, at a time, any two of the following seven quantities (*a* to *g*) as known from the right-hand side of the following equations:

$$x + y = a,\ x - y = b,\ xy = c,\ x^2 + y^2 = d,\ x^2 - y^2 = e,$$
$$x^3 + y^3 = f,\ x^3 - y^3 = g$$

The solutions to fifteen of the twenty-one pairs ($7C_2$) are fairly straightforward while the remaining six are not. This throws light on some of the interesting mathematical contributions of this little known mathematician. His student Tṛkuṭṭaveli Śaṅkara Vāriyar has given a detailed discussion of

these 21 problems and solutions attributed to Citrabhānu in his commentary *Kriyākramakarī* on the *Līlāvatī* (Sarma, VVRI, 1972; pp109 - 125). The *Kriyākramakarī*, one of the major texts of the Kerala School, left unfinished by Śaṅkara Vāriyar was completed by Nārāyaṇa who was also a student of Citrabhānu.

Śaṅkara Vāriyar (A.D 1500–1560) belonged to the Tṛkkaṭṭīri Vāriyar family at Tṛkuṭṭaveli near Ottappalam. As a Vāriyar[20] he was expected to perform certain ceremonial duties at the local temple at *Śrīhutāśa* (*Tṛkuṭṭaveli*). As a disciple of Nīlakaṇṭha, Citrabhānu, Dāmodara and Nārāyaṇa, his astronomical lineage could be directly traced to Mādhava. A patron of Śaṅkara Vāriyar was Nārāyaṇa Āzhvāñceri Tamprākkal, the religious head of Nampūtiris and a well-known promoter of astronomical and mathematical studies. Śaṅkara Vāriyar wrote a concise commentary called *Laghuvivṛti* on the *Tantrasaṅgraha* of Nīlakaṇṭha at the request of his patron (Sarma, 1977; p. 1). This *Laghuvivṛti*, composed in AD 1556, is the last work *of Śaṅkara Vāriyar*. He had earlier written two elaborate and extensive commentaries entitled the *Yuktidīpika* and the *Kriyākalāpa* on the *Tantrasaṅgraha*. However, with hindsight, the most important of his commentaries was the *Kriyākramakarī* on the *Līlāvatī* of Bhāskara II. This work was left unfinished after the verse 199 of the *Līlāvatī* but at the insistence of a well known Kerala literary scholar Mahiṣamaṅgalam Śaṅkaran Nampūtiri; his son Mahiṣamaṅgalam Nārāyaṇa (c. 1500 -1575) took up the task of completing the commentary (Sarma, 1975; p.391). The *Kriyākramakarī* is important in the history of Kerala mathematics for its detailed discussion of various citations from the works of earlier writers (Govindasvāmin, Śrīdhara, Jayadeva, Mādhava among others), some of which are not extant, and for providing detailed rationale and proof of a number of results. Two other works attributed to Śaṅkara Vāriyar are the *Karaṇasāra* and an elaborate commentary on it in Malayalam namely the *Karaṇasārakriyākrama*. The *Karaṇasāra* is an astronomical manual in four chapters composed in A.D 1550.

Another student of Nīlakaṇṭha and Dāmodara and a contemporary of Śaṅkara Vāriyar was a Nampūtiri called Jyeṣṭhadeva (fl. 1500-1610), the author of the seminal text of the Kerala School called the *Yuktibhāṣa*. He belonged to the

Paraṅṅoṭṭu family (*Parakroḍa*) in the Ālathūr village, the birth place as we have seen of a number of earlier Kerala mathematicians. This *Paraṅṅoṭṭu* house still exists in Ālathūr. The *Yuktibhāṣā* is in two parts, with the first part in Malayalam giving a logical and systematic treatment of several important mathematical and astronomical concepts that were in use among the astronomers and mathematicians of that period. It also contains several results of Mādhava pertaining to infinite series for π and sine, computations of sine values, series approximations along with their detailed rationale (Maru Thampuran, 1952; *ch*. vi, vii). The second part of the *Yuktibhāṣā* is in Sanskrit and it is an astronomical treatise which gives rationale of various astronomical results. C.M. Whish ascribes to Jyeṣṭhadeva the authorship of another astronomical treatise *Dṛkkaraṇa* in Malayalam composed in A.D 1608 (Whish, 1835; p.523).

The Malayalam version of the *Yuktibhāṣā* became an important source of dissemination throughout Kerala.[21] Based on *Nīlakaṇṭha*'s *Tantrasaṅgraha*, it is unique in that it gives detailed rationale, proofs or derivations of many theorems and formulae in use among the astronomers/mathematicians of that time.

Jyeṣṭhadeva's student Acyuta Piṣāraṭi refers to his teacher in his work *Uparāgakriyākrama* and indicates that Jyeṣṭhadeva is very old and alive at the time of composition of the work in A.D 1592. A Malayalam commentary confirms that Acyuta's work was based on his interpretations of the instructions that he had received from his very old teacher Jyeṣṭhadeva.

Acyuta came from a community who were not Brahmins but performed external temple functions such as cleaning and supplying of flowers and plants for the temple. They were also employed by some Nampūtiri families to give advanced instructions on the calculation of the astrological calendar (*pañcāṅgam*) and time reckoning.[22] He lived during A.D. 1550-1621 and also came from the epicenter of Kerala mathematics: Tṛkkaṇṭiyūr near Tirur in Ponnani Taluk of South Malabar.

Acyuta was a versatile scholar who made a mark not only in astronomy but in literature and medicine. He attracted the attention of Raja Ravi Varma of Veṭṭathunāḍ (or Venad) and

earned the highest praise for his astronomical skills such that a contemporary, Vasudeva in his *Bhramarasandeśa*, even described him as greater than even Lord *Śiva* in the sense that Lord *Śiva* adorns only one moon from the zodiac on his head whereas Acyuta mind adorns the entire zodiac! (Ulloor II,1970, p. 353). His major contribution is found in his work, *Sphuṭanirṇaya* (composed before A.D 1593), where he introduces for the first time in Indian astronomy, a correction called "Reduction to the ecliptic", around the same time as *Tycho Brahe* (A.D 1546-1601) did in Western astronomy.[23] A rationale for this was provided in his another work *Rāśigolasphuṭānīti*. *Nityānanda* of *Indraprastha* has mentioned about this correction in his work *Siddhāntarāja* in AD 1639 for the first time in the North (Dvivedi, 1933; p.101).

In other works, Acyuta showed his considerable versatility[24], by not only composing standard treatises in astronomy, but also astrological works such as the *Jātakābharaṇapaddhati* and the *Horāsāroccaya* based on Śrīpati's *Jātakapaddhati*. He also wrote a Malayalam commentary on Mādhava's *Veṇvāroha*. His students include scholars like Melpattūr Nārāyaṇa Bhaṭṭa (poet and grammarian) and Tṛpānikkara Poduval (an exponent of *Jyotiṣa*).

In Charles Whish's A.D 1832 paper, which drew the attention of the world for the first time to the existence of Kerala mathematics and astronomy, appears the passage:

> "The author of the *Karanapaddhati* whose grandson is now alive in his 70th year was Putumana Somayaji, a Namputiri Brahmin of Trisivapur (Trichur) in Malabar."

A major work in the dissemination of Kerala mathematics and astronomy not only in Kerala but also in the neighboring areas of present-day Tamilnadu and Andhra Pradesh, *Karaṇapaddhati* was recorded in 1732, about two centuries after Jyeṣthadeva's *Yuktibhāṣa*. The author belonged to the Putumana family of Nampūtiris and came from Śivapuragrāma or Covvaram village (Chandrasekharan, 1956; ch: x, vs.12) near Trichur where a house by the name Putumana of traditional astronomers still exists (Chandrasekharan, 1956; p. xxv).

Karaṇapaddhati is a comprehensive treatise covering Kerala mathematics and astronomy. It has one unusual feature. It follows generally the *Parahita* system and only advocates the *Dṛggaṇita* system in the calculation of eclipses. In ten chapters it explains problems that appear in earlier texts, like the *kuṭṭakāra* approach to solving indeterminate equations or the derivation of implicit values for π and for sine and cosine of angles. While it covers more or less the same ground as the *Yuktibhāṣa*, its non-technical clarity in explaining from the first principles methods of deriving various formulae and construction of tables of astronomical constants meant that it became an important source for commentaries, with two in Malayalam, two in Tamil and one in Sanskrit having been discovered so far. Putumana Somayāji, also wrote an elementary manual, the *Nyāyaratna,* for explaining "astronomical rationale to the dull-witted" and practical texts such as the *Veṇvārohāṣṭaka* for determining the positions of the moon at regular intervals.

After Acyuta, little in the way of original work was done, although the tradition of providing corrections and contributing to the preparation of the astronomical ephemeris for the daily needs of the faithful observers of *muhūrta* and practitioners of *jātaka* continued for a long time. The compilation of the *pañcāṅgam* (five limbs) was periodically subjected to *sphuṭa* (or refinement). About one hundred years after *Karaṇapaddhati* appeared the last of the known texts of the Kerala School. The author of this book, Śaṅkara Varma (A.D. 1800-1838) of *Kaṭattanāḍ* in North Kerala, belonged to the royal family of that area (Sarma, 2001; p.15) and was a contemporary of Charles Whish. His book, *Sadratnamāla*, written in 1823, contains many of the results of the Kerala School, given without the rationale or derivations found in the earlier texts. Whish met him and described him as "a very intelligent man and astute mathematician". He died six years after Whish's article on Kerala mathematics and astronomy appeared in A.D 1832. He was a gifted astrologer who is reported to have predicted astrologically the time of demise of the King and others including his own very accurately (Sarma, 2001; p.4).

This is but only a short account of a vast tradition and as such only a few landmarks on the highway have been simply

touched. Explorative studies have been carried out on a small percentage of the mass of manuscripts that have come down to us from the past. An enormous mass of literature is still lying unexplored in various repositories. A monograph entitled *Science Texts in Sanskrit in the Manuscripts Repositories of Kerala and Tamilnadu* compiled by K V Sarma identifies as many as 3473 science texts in Sanskrit and 12244 science manuscripts from more than 400 repositories in Kerala and Tamilnadu. A discriminating, original and critical study of materials gathered from the past is necessary by which the old knowledge in its depth and fullness can be recovered and restated in modern terms in a faithful and intelligible manner.

DISCUSSION

This paper contains a detailed and comprehensive account of the development of Kerala mathematics, from its original inspiration in Aryabhata in the sixth century to Acyuta Piṣāraṭi in the sixteenth century. The discussion takes place in the religious, social, historical and political contexts that conditioned this long process of development in mathematics and astronomy over a thousand years.

The paper also describes how Madhava's 'passage to infinity' led to developments that anticipated many important European mathematical discoveries in infinite series, analysis and calculus. These discoveries of the Kerala School were driven not by geometrical model-building, as in Western and Arabic astronomical traditions, but by developing more powerful computational techniques. Equally significant is that these computational concerns also led to discoveries anticipating those in European astronomy in the 17^{th} century. The most significant is Nilakantha's adoption of a model for better computation that is similar to the geo-heliocentric theory later proposed by Brahe.

However, the point was raised that paper seemed to excessively focus on the social and political contexts of the Kerala achievements, at the price of losing contact with its mathematical and astronomical context. The achievements of the Kerala School would have been better illuminated if greater attention had been directed towards the intellectual and

computational concerns of the Indian mathematical astronomers. Instead, such 'scientific' issues were in a number of cases delegated to the footnotes.

For example, one note describes the epicycle theory of planetary motion in Kerala astronomy, and the calculation of the true position of a planet through a series of computations involving computing the mean planet and applying corrections to it. Another note refers to how approximations of values using infinite series were improved by correction functions, and even the use of other infinite series to better approximate such functions. These techniques anticipate seminal discoveries in calculus and analysis later made in Europe. Buried in the notes are also accounts of how the Indian astronomers measured and computed planetary positions along the ecliptic, and plotted coordinates from a zero point starting from Lanka. Showing how these Indian mathematical astronomical ideas differ from modern ideas would help, not only to contextualize the Kerala achievements, but also pave the way to showing their relevance to modern science and mathematics.

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ENDNOTES

1 In the Āryabhatan School, which is of fundamental importance to Kerala Astronomy, the epicycle theory of planetary motion is used. For the Sun and the Moon, this is easily explained. The mean planet which is a point that moves with uniform rate of motion in a circle with the earth as centre. The true planet moves in a circle called *mandavṛtta* (epicycle) with the same rate as the mean planet, but in the opposite direction. One can visualize this as the motion in an eccentric circle. In other words, the planet moves in a circle uniformly, but the earth is not at the geometrical centre of the circle, but at a distance equal to the radius of the *mandavṛtta*. The computation of the planet requires the value of R sine of arcs and led to the study of trigonometry and the table of *jyās*. In the case of other planets Mars, Mercury, Jupiter, Venus and Saturn, the *s'īghrvṛtta* is also required. One important feature of Āryabhata School is that the radii of *manda* and *s'īghravṛttas* are not fixed for all the planets except the Sun and the Moon. In *Sūryasiddhānta*, the *mandavṛttas* of the Sun and the Moon are also not fixed and this theory leads to the orbit consisting of two elliptical arcs. This

theory, however, did not catch the attention of the Kerala Astronomers.

2 They include scholars such as C. Srinivasaiengar, Rama Varma Māru Thampuran, Akhilesvara Aiyer, C.T. Rājagopal, Vedamurthy Aiyer, A. Venkataraman, M.S. Rangachari, T.S Kuppanna Sastri, Balagangadharan, K. Kunjunni Rāja, T.A. Sarasvati Amma, P. K. Koru, R. C. Gupta, A.K. Bag, K.S. Shukla, and K.V. Sarma. The knowledge excavated consists mainly of Parameśvara's formula (14[th] century) for circum radius of a cyclic quadrilateral in terms of its sides (now known as Lhuiller's formula, first stated in 1782); the so-called Leibniz series for π, the Gregory series for inverse tangent and several other slowly and rapidly convergent series for sine, cosine and others, anticipated by Mādhava in 14th century and elaborated by his disciples; the concept of approximation of infinite series using remainder terms; estimation and analysis of error in truncation of infinite series using correction functions; derivation of other infinite series from the error functions so obtained and several geometrical and analytical rationale of various results by employing sophisticated mathematical techniques reminding one of modern calculus and analysis.

3 See Ulloor S. Paramesara Aiyar (1953), Volume I, p. 165

4 *Candravākyas* of Vararuci can be regarded as one of the earliest contributions to Astronomy. The vākyas are helpful in finding the Moon by the method of extrapolation. This can be explained in the following way. Record the positions of the planet on every day at a specified time (say, the sunrise at Laṅkā, the zero position) for a long period till the planet comes back to the original position nearby. The table so prepared can be used for the subsequent periods by making the correction on account of the difference in position in the end of the table. *Candravākyas* gave the position of the Moon for 248 days corresponding to nine anomalistic periods. For finding the position of the Moon at sunrise at Laṅkā find the *vākyas* to be used and make the correction called *dhruva*. One can get the position for any place at the sunrise by making the corrections for the difference in the sunrise (*cara*) and longitudinal difference (*des'antara*).

5 In the system of word-numerals known as *Bhūtasāṅkhyā*, numbers were indicated by well-known objects or ideas commonly associated with the numbers. Thus, zero was represented by *śūnya* (void) or *ambara, ākāśa* ('heavenly space' probably meaning ether) or *kha*(sky) etc., one by *rupa, Indu* (moon) or *bhū* (earth) etc., two

by *netra* (eyes) or *paksha* (waxing and waning of the moon) or *bāhu* (hands, shoulders), *Aśvin* (*twin gods*),*etc*., three by *guṇa*(the three qualities), *agni* (the three sacrificial fires; also the three tongues of fire), *kāla* (time: past, present and future) or *loka* (heaven, earth and hell) etc., four by *veda, yuga*, etc., and so on. With multiple words available for each number, the choice of a particular word for a number would be dictated by literary considerations. This form of notation continued for many years in both secular and religious writings because it was aesthetically pleasing and offered an easier way of remembering numbers and rules.

According to the *kaṭapayādi* system of word-numerals, the letters *ka, ṭa, pa* and *ya* of Sanskrit alphabets represent 1; the letters *kha, ṭha, pha* and *ra* denote 2 etc as shown in the table below. For details see Madhavan (1991).

ka	*kha*	*ga*	*gha*	*na*	*ca*	*cha*	*ja*	*jha*	*ña*
ṭa	*ṭha*	*ḍa*	*ḍha*	*ṇa*	*ta*	*tha*	*da*	*dha*	*na*
pa	*pha*	*ba*	*bha*	*ma*					
ya	*ra*	*la*	*va*	*śa*	*ṣa*	*sa*	*ha*	*ḷa*	*vowels*
1	2	3	4	5	6	7	8	9	0

This system could equally apply to other languages and its application to Malayalam became widespread. If such a system is applied to English, the letter *b, c, d, f, g, h, j, k, l, m* could represent numbers 0 to 9. So would *n, p, q, r, s, t, v, w, x, y*. Assume that the remaining letter *z* represents 0. The vowels, *a, e, i, o, u* serve the function of helping to form meaningful words. Thus the old word, "Madras", would be represented by 9234 and *"love"* by 86 read in the usual manner. A system devised to facilitate memorisation, since for any particular number, different memorable words could be made up with different chronograms.

6 For a detailed discussion of different methods of calculating planetary positions, including using *Parahita* and *Dṛggaṇita* systems, see Sarma (1972 Hoshiarpur).

7 *Āryabhaṭa*'s complicated alphabetical system of notation is based on dividing 18 places of figures into 9 pairs, each pair consisting of a *varga sthāna* (i.e., a square place or odd place: like a unit's place, hundred's place, ten thousand's place) and an *avarga sthāna* (i.e., a non square place or even place: like those of 10, 1000, etc.); with the nine *varga-avarga* pairs denoted by the nine vowels (*a, i, u, ṛ, ḷ, e, o, ai, au*) successively. The odd and even positions are

determined by the nature of the consonants associated with the vowels. The consonants from *ka* to *ma* having values from 1 to 25 respectively are *varga* consonants and those from *ya* to *ha* having values from 3 to 10 respectively are *avarga* s. The varga consonants are to be placed in the odd positions and the *avarga* s in the even positions.

8 A *bhāṣya* is more than a commentary, for apart from an exhaustive survey of relevant literature and a study of the text, it often served as a vehicle to initiate original investigation on the same and related topics.

9 *Govindasvāmi*'s procedure is an extension of *Brahmagupta*'s interpolation procedure from his book *Khaṇḍakhādyaka* (AD 665). For further details, see Gupta (1969).

10 The *Bhaṭṭatiris*, a sub-group of Nampūtiri Brahmins, became the custodians of Vedic knowledge. They performed all the duties associated with the sacred fires (*agnihotris*), except the performance of sacrifices. After an intensive study of the Vedas, they were required to learn logic (*tarka*), grammar (*vyākaraṇa*), religious philosophy and rituals (*vedānta*, *mīmāmsa*), etc and then teach these disciplines at the *śālas* attached to the temples.

11 This connection is supposed to date back to a night which he spent with a woman who belonged to the *Kaṇiyān* family at *Pāzhūr* in the Vaikom Taluk. The progeny of their union is the ancestor of this famous family of astrologers.

12 Mādhava of Saṅgamagrāma founded a School that had the following teacher-student lineage:
Mādhava (fl.1340-1420) ==> Parameśvara (fl.1380-1460) ==> Dāmodara (fl.1450) ==> Nīlakaṇṭha (b.1444) ==> Chitrabhānu (fl.1530) ====> Nārāyaṇa (fl. 1529) and Śa'nkara Vāriyar (fl. 1556) Also Dāmodara ==> Jyeṣṭadeva (fl.1500-1575) ==> Acyuta Piṣāraṭi (fl.ca.1550, d. 1621).

13 A manuscript says: "*Mādhavan Veṇvārohādīnām karthā... Mādhavan Ilaññipalli Emprān*". For details see K.V. Sarma's paper, 'Some direct lines of astronomical tradition in Kerala' P.C.Shastry Felicitation vol, Delhi, 1972.

14 Other available works of Mādhava are the *Madhyamānayanaprakaraṇa, Mahājyānayanaprakāra, Lagnaprakaraṇa, Agaṇita, Agaṇitapañcāṅga, Agaṇitagrahacāra.* A text entitled *Golavāda* attributed to Mādhava secured him the appellation *Golavid* or *Golatattvavid*.

15 These included the authors of the major publications such as Nīlakaṇṭha, Jyeṣṭadeva, Nārāyaṇa and Śaṅkara Vāriyar, and Putumana Somayāji. The attribution to Mādhava usually took the form of either a statement such as "*.... atrāha Mādhavaḥ...* ", or "*uktam ca Mādhavena*", or *Saṅgamagrāmajo Mādhavo'pyevamāha..* " etc. ("... thus said Mādhava...."etc.) preceding quotations of results or statements such as "*.. Iti jyācāpayaḥ kārya grahanam Mādhavoditam*" (... computation of the arc from the sine is given by *Mādhava* "). There is every likelihood that Mādhava wrote a treatise on mathematics and astronomy from which quotations were given by later writers. Unfortunately, such a treatise is no longer extant. or not yet unearthed.

16 In the verse 91 of chapter IV *Paramesvara* states that he resides in a village 18 *yojana-s* to the west of the central (i.e., Ujjain) meridian, having sine latitude 64° 7' .

17 Colophon to *Laghurāmāyaṇam* of Rama gives these personal details about Nīlakaṇṭha. (For identification of this Rama's father Nīlakaṇṭha with the astronomer Nīlakaṇṭha Somayāji see Menon, 1952 - 1953)

18 The discussion that follows on Nīlakaṇṭha's contribution is based on Ramasubramanian *et al.* (1994) to which reference should be made for details.

19 According to one account, Nīlakaṇṭha was born on June 17, 1444 and died in 1545.

20 The *Vāriyar-s* (or *pāraśava-s*) in Kerala form a class of non-Brahmin temple officials who assisted the Brahmin priests in their religious rituals as external functionaries. A number of them were skilled in astrology and many were learned in Sanskrit. By birth Śaṅkara was a *pāraśava* and so by profession he was an external functionary of the temple at *Śrīhutāśa* (See, Sarma, 1975; p. 391)

21 There are close similarities between this text and Śaṅkara Vāriyar's (and Nārāyaṇa's) *Kriyākramakarī* and *Yuktidīpika*, where the former is a *bhāṣya* on the *Līlāvatī* and the latter on the *Tantrasaṅgraha*. However, Śaṅkara Vāriyar acknowledges the source of some material in the latter text is the work of Jyeṣṭadeva (who is referred to as the Brahmin of *Parakroda*).

22 There were two common methods of telling time. At a point early in a child's education, the two methods were taught in the form of verses which were to be memorised. The *ativakyam* showed how to tell the time of the day by measuring the length of the shadow before and after noon. The *nakhstvakyam* showed how to reckon time at night by the

position of stars and particularly by the time at which certain stars rose. This required considerable knowledge of astronomy and hence the method was only sketched out with further elaboration at an older age. At a later age, an *acārya* (teacher), usually a *Piṣāraṭi* gave them further instruction on the use of water clocks for time reckoning where the basic unit of time was *nāḍika* (24 minutes) or the time that a typical vessel took to sink.

23 In astronomical calculations, the longitude of a planet is measured along the ecliptic, while, in fact, its motion takes place along its own orbit, which deviates slightly from the ecliptic. For an accurate computation of the planet's position, this deviation has to be corrected. Acyuta Piṣāraṭi gave the following formula (expressed here in modern notation) for the correction in the case of the moon:

$$R = \frac{\sin\alpha\cos\alpha(1-\cos\beta)}{\cos\theta}$$

where α is the longitudinal difference between the node and the planet, β the maximum latitude and θ the actual latitude.

24 His versatility is evident from the range of his compositions which included calculation manuals such as *Sphuṭanirṇaya Tantra* (dealing with true computation of planets); the *Rāśigolasphuṭānīti* (dealing with true longitude computation on the zodiac sphere); the *Chāyāṣṭaka* (dealing with computation of gnomonic shadow); his exposition of the *Dṛggaṇita* system in *Karaṇottama*; the computation of eclipses called the *Uparāgakriyākrama* or *Uparāgaviṁśati*; a commentary on the *Suryasiddhānta* and a commentary on Mādhava's *Veṇvāroha*.

3

AN INTELLECTUAL HISTORY OF MEDIEVAL KERALA
- With special reference to Mathematics and Astronomy -

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The medieval culture of Kerala had its internal contradictions. On the one hand, Kerala's contacts with the outside world through trade and commerce increased substantially, linking her with the contemporary world economic system. On the other, the particularities of local cultures, such as the localisation of weight and measures[1] and the prevalence of other local customs and practices, continued to hold sway.[2] Again, while at the same time that medieval Kerala witnessed a remarkable development of science and technology, social iniquities such as the *Jenmi* system[3] became deeply entrenched in the social fabric of the state.

The medieval period of Kerala stretched over many centuries from the age of the Perumals of Mahodayapuram (9^{th} to 12^{th} centuries) to the formation and establishment of the *swarupam*[4] organizations (13th to 17th centuries). This period was marked by various historical developments, such as the spread of agricultural and village communities, a phenomenal rise in local as well as overseas trade, and the increasing struggle among the *swarupams*

themselves to become the dominant polity. It was during this period that foundations were laid for what came to be known later as the Kerala School of Mathematics and Astronomy[5].

An agrarian society set within a monsoon region, needs to know how to predict seasonal and climatic changes. For that an accurate calendar and astronomical computations regarding the position and movements of celestial bodies are essential. Earlier Sanskrit works on these topics were studied and translated, and commentaries, written in Sanskrit or Malayalam, elaborated and extended these works.[6] An example would be a 14th century text on astronomy by a major figure of the Kerala School, Vattasseri Parameswaran became a reference guide for enabling efficient cultivation of both paddy fields and dry land.

Astrology originated in Kerala to fulfil certain needs for finding out auspicious times for certain rituals and observances. Auspicious times were sought and calculated for the naming ceremony and first ceremonial feeding of an infant, initiation during childhood, marriage etc. It was also necessary in the compilation of a *Jataka* (horoscope) to foretell the course of life of a person based on the study of the position of certain planets and stars at the time of birth.

The advent of new ideas into medieval Kerala culture came from two different sources— from the pan Indian tradition and from overseas[7]. The Nambuthiris, the Jains and the Buddhist were responsible in different ways for the introduction of Sanskritic culture. Sanskrit texts containing these traditions originated from different regions of the sub- continent and not all necessarily from the North. During the early part of the medieval period, great importance was given to the study of these texts and attempts made to present a synthesis of the knowledge gathered in regional languages.

The incorporation of new ideas into local cultures affected many disciplines, including literature, religion, art, science and technology, and helped the shaping of a peculiar regional identity for Kerala. The attempt at borrowing elements of a pan Indian tradition resulted in a surprisingly vast output in scholarly literature. The texts produced were in Sanskrit, in *Manipravalam* (a mixed language of Sanskrit and Malayalam) and in the emerging new language of Malayalam. Although elements of

this pan Indian tradition had various regional representations, the scientific knowledge emanating from the Sanskritic tradition was confined more or less to the upper strata of society, while its practical aspects (including physical sciences and medicine) became the preserve of the lower strata or of all. Astrology was handled by the *Ganaka* community.[8] Medical science was practiced by the almost all sections of society. For example, there were eminent physicians found even among the lower caste Vela or Mannan community.

The early origins of Kerala astronomy is associated with the legendary fourth century figure of Vararuchi who composed the *Chandravakya* ('moon sentences') which was helpful for astronomical calculations.[9] The *Kadapayathi* system of numeration and the *Parahitha*[10] system were also some of the earliest aids to the study of mathematics and astronomy. The *Parahita* system was promulgated by Haridatta, in the 7th century as a correction to Aryabathan computational system. In his *Grahacharanibandhana*; Haridatta used the *Kadapayathi* system even though it was known by another name until the 9th century when Govindaswami wrote the *bhasya* on *Mahabhaskariya*, an important text in the development of Kerala mathematics.

In the 9th century Sankaranarayana, the court astronomer of Sthana Ravi Varma wrote a commentary on the *Laghubhaskariya*[11]. During that period at least in the capital city of the Perumals, Mahodayapuram, arrangements were made for studying astronomy. Sanakaranarayana refers to the Observatory at Mahodayapuram even though no archaeological evidence has been found with regard to location or presence of the Observatory. There were also reports of a public announcement of the correct time at regular intervals.[12]

The kind of an intellectual atmosphere that prevailed at the time may be gathered from references in the *Kokasandesam*, a 15th century text written in *Manipuravalam*. *Kokasandesam* refers to a great teacher (or preceptor) and his students (*sisya*) at Triprangode.[13] There is also a reference to a long-established practice of announcing time at regular intervals in Triunavaya for Brahmin scholars.[14] Sankaranarayana was also a disciple of Govindaswami. Govindaswami and Sankaranarayana are referred

to in a 11th century text, *Udayadivakara*, the first as the author of an elaborate commentary on *Laghubhaskariya* and the latter as an outstanding scholar of *jyothisha* (astronomjy/astrology) *ayurveda* (indigenous medicine)[15] and *tantras* (metaphysics).

Among the outstanding astrologers of the period was Govinda Bhattathiri of Talakulam who lived in the 13th century. His birth place has been identified Alathiyur in the present Malappuram District. The shrine of his tutelary deity is still preserved at Alathiyur. He lived between 1237 and 1295. His *Muhurthapadavi* formed the basis for several later works under this name. He made elaborate commentaries called *Dasadhyay* on the *Hora*. His *Muhurtharatna* remained very popular in Kerala for many years.

Preceding him was a list of notable astrologers that included Keralacharya (a versatile scholar of *jyothisha, ayurveda* and *tantra*) who served under King Rama Varma (1108-1131); Vyaghrapada, a native of Vykom in central Kerala and author of an influential treatise on analytical astrology; Krishnacharya, the author of *Chintajnana*, a comprehensive work on astrology; and Vidhya Madhava and his son Vishnu of Neelamana , both astrologers of note.

While the Jains and the Buddhists were earlier instruments in the dissemination of Sanskritic traditions from the North, it is generally accepted that the coming of the Nambuthiris marked a turning point in the development of science in Kerala. The Nambuthiris came and settled in the fertile river valleys of Kerala. According to tradition there were 32 Brahmin and almost all of them have been identified.[16] It is in and around these settlements that the mathematics and astronomy flourished in Kerala during the medieval period.[17] There were settlements in Alathiyur and in nearby places of Trikandiyur, Tripangode, and Thirunavaya in the Malappuram District and Perumanam and Iringalakuda in Trichur District.[18] The Nambuthiris, because of their superior knowledge of science and technology and their high ritual status, established their dominance over the indigenous people. However, they did not attempt to impose their language on the people and soon became well versed in the local languages of Malayalam. It was the Nambuthiris and the Ambalavasi[19] communities of Variyars and Pisharotis who

contributed substantially to the development of Kerala mathematics and astronomy.

There is a belief that the Nambuthiri contributed to the general moral degeneration during the late medieval period in Kerala. However, only a small minority of the Nambuthiris lived a life of idleness and self-indulgence. The Nambuthris followed a system of patrilineal descent (*makkatayam*), with an unusual form of primogeniture that allowed only the eldest son to inherit land and property and marry Nambuthri women. For the younger sons (or the *aphans*), only a form of concubinage (or *sambandham*) was available usually between them and Nair women. The children of such unions becoming the sole responsibility of their mother's family. Thus, the argument goes, freed from the responsibilities of managing the family properties and family life, the younger sons must have got more time to pursue intellectual and other activities. An interesting question that arises is whether the Nambuthiri members of the Kerala School were younger sons. There isn't sufficient information available at present to answer this question.

During the period under study there prevailed the *Gurukula* system of education, a system found in Kerala for at least a thousand years.[20] Under this system, a student after completing his early education at home was accepted into a *Gurukala* after evaluation by the Guru. Once accepted, the student became a member of Guru's family. The student would then follow the instructions of the Guru, both serving and obeying him implicitly. This learning process involved three stages: learning from the teacher through oral instruction leading to a guru-student bond (*gurusishya parambara)*; learning through participation in discussions (*parishads*) where the great masters presented and discussed their findings; and through a type of meeting (*vidvat sadas*) where scholars presented effective arguments and counter arguments on various issues.

The celebrated *Gurukulas* of Kerala were the Trichur Brahmaswam Matham, Thirunavaya Samuha Matham, Kudallur Mana Gurukulam, Thiruvalla *sala*, Moozhikulam sala and later Kudungallur and Punnasseri Gurukulam. Thiruvalla was noted for instructions in arms to the Nambuthiri students. The Kudallur mana Gurukulam was noted for learning in grammar. There is a

shrine of the great Indian grammarian, Patanjali, at Kudallur Mana, perhaps the only shrine to Patanjali that survives in India today. It should be noted that in the *Gurukulam* boarding and lodging and tuition were free. Simple living and high thinking was greatly encouraged for it was believed that greed and pursuit of wealth would lead to self destruction.

Contemporary literary competitions like *Revathipattathanam* at Kozhikode and *Kadavallur Anyonyam* at Kadavallur (near Kunnamkulam) were evidence of the literary and intellectual liveliness of the period. Students from the *Gurukulams* mentioned above were expected to participate in these competitions. Uddanda Satrikal, the famous Sanskrit poet, once visited Kudallur Gurukulam and was highly impressed by the standard of the curriculum and the tradition of deep scholarship found there.

An analysis of the development of science and technology in medieval Kerala leads on to the following conclusions. The scientific knowledge emanating from the Sanskrit tradition was confined more or less to the upper strata of society whereas technological aspects were open to the wider society. A scholar generally tended to be versatile, with interest in a number of disciplines. The very same person could be a physician, an astronomer or a mathematician at the same time. Among the subject taught in the medieval institution such as the *Gurukula* were grammar, philosophy, medical science, mathematics and astronomy. Specialisation was possible only at a later stage.[21] It is interesting in this context that the Nambuthiris did not practice astrology for any pecuniary purposes. Instead they taught it to others and advised them to follow the right path. We have the classical example to this in the exhortations of Parameswara of Vatasreni to the *Ganakas* to take into consideration the position of planet at sight and not depend on the earlier studies in the same.[22]

The 14th and 15th centuries saw the emergence of the major figures of the Kerala School like Madhava of Sangama Grama, Parameswara of Vatasreni and, Nilakantha Somayaji. They came from the medieval Brahmin settlements mentioned earlier. Madhava of Sangama Grama was known as the *Golavid* (master of the sphere). His house name was Elanhipalli and he came from Dringalakuda, one of the famous Brahmin

settlements. The location of his house, which no longer exists, is at Iringarapalli Mana, near the railway station at Kallettumkara in Iringalakuda.[23] The temple where Madhava was reputed to have spend many hours in meditation is now a private shrine under the management of the Iringarapalli Mana. Two stone slabs about 7 feet long and two and a half feet wide are found on the right side of the temple. It is said that Madhavan used to rest on them for long hours, lost in thought. There is a local belief that Madhava died of tuberculosis and left no heir.

Madhava's disciple Parameswara of Vadasseri, a major figure of the Kerala School, who revised the *Parahita* system of computation, discussed earlier, and introduced his *Drgganitha* system in 1430.[24] He also came from Alathiyur, the Brahmin settlement, mentioned earlier which was also noted for a family of physicians called Alathiyur Nambis. He spent long hours observing stars and planets on the sandy banks of Bharata Puzha and was reputed to have done so for fifty five years.[25] He observed a large number of eclipses, which are documented in his *Sidhantha Deepika*. Parameswara composed about thirty works on astronomy and astrology, including both original treatises as well as commentaries on earlier works. His notable compositions include *Dragganita, Goladweepika* and three texts on improving astronomical computations and explaining the rationale of eclipses. He lived between 1360 – 1455.

Paremeswara's son Damodara was also a well-known astronomer of his time and mentioned often by his distinguished student Nilakanta Somayaji (1444-1545). Nilakantha came from Trikandiyur near Alathiyur. Unlike other astronomers in a colophon to his *bhashya* on *Aryabhatiya* he gives particulars about himself. His family name was Kelallur and father was Jataveda. His brother Shankara was also a noted astronomer. Nilakantha's best-known work is *Tantrasamgraha*. The first and last verses of *Tantrasamgraha* contain the chronograms giving two dates between which the text was composed:

"He Vishno Nihitam Krtsana" (indicating the Kali date 4601, Meenam 26) and

"Lakshmi Sanihita dhyana" (indicating Kali date 4602 Medam 1)

This would imply that he completed *Tantrasamgraha* in five days!

Trikudaveli Sankara Variyar (fl. 1500-1560) came from Trikadiri near Ottapalam in Palakkad District. From a colophon of *Laghuvivriti,* a commentary on *Tantrasamgraha,* we get personal details about Sankara Variyar. His contemporary, Jyeshtadeva, the author of *Yuktibhasha* was a student of Damodara and belonged to the Paranothu family in Alathiyur. Yet another versatile scholar was Achuta Pisharadi (1550-1621). He came from Trikandiyur near Alathiyur. He was patronised by the king Ravi Varma of Vettathunadu and was the teacher of Malapathur Narayana Bhattathiri. The date of death of Achutha Pisharadi is contained in the *Charamasloka* composed by Malpathry. The location of Achuta's house was near the Trikandiyur Siva temple. There one can still see the large trunk of the *Asoka* tree under which he was reputed to have sat in contemplation.

Among the discovered works on mathematics and astronomy in Kerala and on which there are references there is a significant class of texts relating to the rationale of certain method or procedure. These text have words such as *yukti, nyaya* as part of their title. These show great emphasis placed on justifying the methodology and willingness to change or modify procedures when necessary. The works devoted to the demonstration of procedure only have the words *Kriya* or *Kriyakarma* attached to them.

The astronomers of Kerala did not claim infallibility regarding their procedures. They emphasized the need for subsequent observations and experimentation and periodical revisions and corrections. Nilakantha led the way by stating that experimentation and revisions should be continued by successive generations of scientists. Quoting from *Thantaravarthika Meemamsa Suthra* Nilakantha writes:

"Iti vartika karopi graham
gati jnanam anumanaha"[26]

[i.e., knowledge of motion of planets is based on inference. In fact one has to look for a system that tallies (inference) with observation]

In *Pareekshasuthra,* Nilakantha Somayaji provides practical advice on the need for accurate observations and verification of various astronomical phenomena. He suggests that the result of experiments should not be revealed to the students because it would destroy their inquisitiveness. So instead of recording the results arrived at, the methods adopted and the data collected should be recorded.[27]

It is interesting that in the long line of the members of the Kerala School of mathematics and astronomy discussed there wasn't a single Nair. Almost all of them belonged to the Nambuthiri community or the *Ambalavasis.* The reason for this is found in the peculiar *Jati Dharma* prevailing in Kerala which prescribed the duty of *dharma* to each community and which in its turn determined one's own choice of one's vocation. and profession. According to this, the Nairs were to serve in the military service of the rajas and chieftains. Thus even up to the emergence of European control in Kerala, the existing social set up was structured according to the *Jati maryada* and *acarams* (caste customs and practices). No low caste people are found in the group of astronomers and mathematicians. They were engaged in the daily struggle to eke out a living and had no time for intellectual pursuits like mathematics and astronomy.

All the outstanding scientists hailed from Brahmin settlements, especially from the settlements of Alathiyur and its surroundings. The village of Alathiyur has supplied many an incredible number of astronomers, mathematicians, physicians and other scientists and witnessed a long chain of *gurusisya parambara.* These scholars and their remarkable contributions have generally been ignored by the historians of Kerala and hardly known to the outside world.

Another interesting aspect of these scholars was their intimate connection with the local temples. The Temple was the centre of socio-economic and cultural life of medieval Kerala. It was all the more so with regard to intellectual life. Scientists and men of letters associated with the Temple were not so much temple functionaries but as devotees at the shrine of the tutelary deities. The Bhattathiris who figure among the mathematicians, astronomers and astrologers had a closer association with the temples than others. They were employed

in the temple as *Bhattas* or Vedic scholars of the *Puranas*. The Tiruvalla inscriptions refers to such scholars. The remuneration they received was known as *'Ma parata Virut'*. Thus, the astronomer-mathematicians and other men of letters of this period were closely associated with temples. Almost all of them had tutelary deities. And they often remembered them by introducing a colophon of their works.

An examination of the Settlement Registers of the areas connected with the scholars and scientists show that the bulk of the cultivated lands in the area belonged to the Nambuthiris and the temple managements. It remained so even at the beginning of 20th century. This indicates that these scientists had a material base for engaging themselves in the pursuit of knowledge. The family of properties of a number of these scientists were merged with those of other families either because of lack of heirs or because of them being sold. So the final destination of the properties need not have been the Temple or the Nambuthiri community. A good example of this is the case of Etakramanancheri Nambuthiri (1625-1700), the author of *Bhadradipa Ganita*. A few generations ago the compound of the *illam* (residence) of this astronomer came into the hands of a Nair family. Now the members of that family add the suffix *Etakramanachery* to their names. Sometimes two or more families would merge together when there is no heirs. This is illustrated by the Desamangalam Mana at Vatakkanchery, Thrissur district, where the family members use five alphabets (AKTKM) as prefix to their names denoting the five families that were included in their Mana.

Caste had a major impact on the development of astronomy and mathematics. Almost all the outstanding astronomers and mathematicians belonged to the upper strata of the society, notably the Nambuthiri and Ambalavasi communities. It was not because such form of knowledge were forbidden to the lower caste but because the imparting of education took place inside the residence of the guru to which the low caste people had no access. Sankara Varier and Achutha Pisharoti as Ambalavasis were considered high-caste although they were not Brahmins. Traditionally they were in charge or flowers and garlands for the temple and were therefore in close association with the Brahmin

priests. Their tasks within the temple gave them ample time to engage in intellectual pursuits. The language of Sanskrit was not forbidden to non-Brahmins. What were denied to them the recitals of the Vedas and the performance of rituals? Astronomers like Achutha Pisharoti and Sankara Varier were also Sanskrit scholars.

The Nambuthiri were well versed in Sanskrit. But they did not impose their language on the people under their control, like the later colonial masters. They accepted Malayalam as their language and called it *bhasha*. All the Sanskrit words used by the Nambuthiri were automatically translated into Malayalam. For example *Somayaji* became *Chomatiri, Samavartanam, Chomartam* and so on.

Yukti Bhasha are written on Malayalam. It had a practical purposes. Apart from the intellectual pleasure derived from writing a book, it was exigencies of needs that made them compile treatises. Thus even the works of the illustrious astronomer Sangamagrama Madhava may be seen as a response to the needs of agriculture. There is nothing to indicate that astronomy and mathematics were considered as low-status disciplines. They flourished along with *Ayurveda*, which enjoyed a popularity wider than the two disciplines mentioned. This may have been due to the discipline of *Ayurveda* being open to a wider section of the community compared to the other two. But the main reason may have been the specialised pre-requisites required for the study of astronomy.

ENDNOTES

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1 See Vijayalekshmy, M. "Localism of weights and measures in Precolonial Kerala." *Proceedings of Indian History Congress*, Bangalore, 2004.

2 The entire age was marked by a regionalisation of culture and the emergence of what may be described as the 'Kerala identity'. This is best manifested in the emergence of the regional languages of Malayalam. In the 7th and 8th centuries there was a major transition from *tinai* (scattered homesteads) to village settlements. The emergence of village settlements was the beginning of the regionlisation. For further details, see M.R. Raghava Varier, *Village Community in Pre-Colonial Kerala*, Delhi, 1994.

3 Under the *Jenmi* system, a landlord exercised not only absolute proprietorship of land and the resulting powers to evict tenants from his land at will but even possessed the power of life and death over them. A system of agrarian serfdom came into existence where the sale of any land meant that its tenants and workers followed as chattels.

4 *Swarupams* were ruling families which came to control the *Nadu* divisions. They were large extended families whose political authority was organised on *kuru (*or order of seniority). For further details, see M.R. Raghava Varier, "State as Swarupam: An Introductory Essay", R. Champakalakshmi, et al. (ed.), *State and Society in Pre-Modern South India*, Trissur, 2002, pp.120-130.

5 Madhava of Sangamagramma began a School that had the following teacher-student lineage:
<u>Madhava</u> (fl.1340-1420) ==><u>Parameswara</u> (fl.1380-1460) ==> Damodara (fl.1450) ==> <u>Nilakantha</u> (b.1444) ==> Chitrabhanu (fl.1530) ====> <u>Sankara Variyar</u> (fl. 1556)
Also
Damodara ==><u>Jyesthadeva</u> (fl.1500-1575)==> <u>Acyuta Pisharoti</u> (fl.ca.1550, d. 1621)
The names underlined are generally recognised as the major figures of the Kerala School.

6 About 400 palm leaf manuscripts on astronomy and 350 on astrology have been discovered in Kerala. Many remain hidden from public view due to the conservative attitude of the custodians of these manuscripts. In the hot and wet climate of Kerala a number have perished or are in such a poor state that it is impossible to decipher them. Certain areas of Kerala, especially in Malappuram and Trichur districts, were the centres of mathematical and astronomical activities during the medieval period. These areas have not been sufficiently trawled for new sources of manuscripts. (See the various censues of K V Sarma, starting with his *History of Kerala School of Hindu Astronomy*, Hosharpur, 1972, p.viii)

7 The influence of alien cultures coming in the wake of overseas trade led to the formation of trade and cultural diasporas along Kerala coast. The settlement was a window to foreign culture where foreign culture was manifested through food, dress, trade mechanism etc. For details of trade diasporas see Philip de Curtin, *Cross Cultural Trade in World History,* London, 1984, pp 10-11 and Abner Kohen, "Cultural Strategies in the Organization Trading Diasporas", Claud Meillassoux, (Ed), *The Development of Indigenous Trade and Markets in West Africa,* London, 1971, p. 267.

8 Some of the *ganakas* (or traditional astrologers who came from a low caste group) were highly respected during this period in providing auspicious times for holding court festivals, undertaking state activities and even declaring wars. For example, one of the most popular texts was Rana Deepika by Kumaraganaka (14th–15th century) who enjoyed the patronage of Deva Sarma a prince of Ambalappuzha Royal family. For further details, see Ulloor S. Parameswara Iyyer, *Kerala Sahithya Charithram*, Vol. II, Thiruvananthapuram, 1953, pp. 112-13.

9 Through a series of "nonsense" mnemonic words or phrases, the positions of the moon at regular intervals each day was traced to help worshippers carry out their daily observances and rituals.

10 *Parahita* literally means 'desired by others' or 'suited to others'. The system is a description of certain corrections to the Aryabhatan planetary system introduced by Haridatta. These included *Sakabda Samskara* or *Bhata Samskara*. The *Parahita* system became the cornerstone in the propagation and practice of astronomy in Kerala.

11 *Laghubhaskariya* is a commentary by Bhaskara I (c. AD 600) on *Aryabhatiya* of Aryabhata. This is a text which had a considerable influence on Kerala mathematicians

12 Sankaranarayana refers to the arrangements for announcing the correct time to the public at regular intervals of *ghatikas* (24 minutes). See P.K Narayana Pilla (ed), *Laghu Bhaskareeya Vyakha of Sankara Narayana,* Manuscript library, Thiruvananthpuram, 1949, Chapter III, *Sloka* 18, 20 and 22.

13 See Elamkulam Kunjan Pillai (ed), *Kokasandesam, Kottayam,* 1959, Verse 17, p. 36.

14 Ibid Verse 22, p.40.

15 The science of *ayurveda* owes its origins in Kerala to Buddhist traditions and texts like *Ashtanga Hrdaya* and *Ashtanga Samgraha.*(Book Nos.121, 238c, 1347 and 2395A, Manuscript Library, University of Calicut) The practitioners of this system of medicine did not come from any specific caste. In veterinary science, an allied system to *ayurveda*, great importance was given to the treatment of elephants as shown by the widely popular texts like *Gaja Chikitsa, Gaja Chikitsanidhi*, etc. (Book 559, Gaja Chikitsa, Book No. 500C, Gaja Chikitsavidhi, etc., Manuscript Library, University of Calicut)

[16] Herman Gundart (ed), *Keralopathi* (1843), Balan Publications. Thiruvananthapuram, p.4. Also see Kesavan Veluthat, *Brahmin Settlement in Kerala*, Calicut, 1978.

[17] Some of the more important astronomical computations included tracing the movement of the moon. The fast motion of the moon amounting to 13 degrees a day and consequent quick change in its position during day required special consideration on its correct computation. Madhava of Sangama Grama arrived at accurate moon-mnemonics correct to seconds. Kerala Astronomers made intense investigation in the computation of eclipses. Works of Narayana, Achuta Pisharadi and Parameswara reveals this. The *Grahana Mandana* of Neelakanta Somayaji is an elaborate discussion on the tradition and rationale of computing eclipse. There were shadow computations also. Regarding the matter of preparing annual almanacs the general trend has been to prepare them for one year at a time. But Shankaran Nambuthiri of Mahisha Mangalam calculated *Muhurthas* for a thousand years.

[18] The largest number of astronomers and mathematicians from Alathiyur were *Rgvedins* for whom the study of astronomy and the compilation of calendar were essential tasks since they had to determine the auspicious time for the performance of the *yagas*. The *yagas* and their ritual importance enhanced the social status of the Nambuthiries. It also added to their material wealth since each Nambuthiri who performed a *yaga* was entitled to a share of the corporate wealth called *Padakaram*. For further details, see M.R Raghava Varier, "The Nambuthiries Ritual Tradition with Special Reference to Kollangode Archives", Staal (ed), *Agni*, Vol. II, Berkeley, 1983, pp. 291-292.

[19] The *ambalavasis* or *antaralajatis* (intermediate caste) consisted of a sub-caste of temple servants. Probably as a result of their close association with the Brahmins in the temple, there emerged among them scholars of Sanskrit, physicians, astronomers and poets who were to rival the Nambuthris themselves.

[20] *Gurukulas* developed in the families of aristocratic Brahmins, palaces of the chieftains and even in the houses of ordinary man who could afford to maintain it. They emerged between the *salais* of the early medieval period and the collegiate system of the modern period.

[21] It is important in this context to examine what was happening on the doorstep of Kerala. The Lakshadweep people with their knowledge of Arab astronomy and their knowledge and acquaintance with the indigenous Indian astronomical traditions developed a highly

scientific lore of navigation called *Rahmani*. This system divided the eastern horizon into sixteen parts and developed a method of estimating the distance between the islands on one hand and between the port towns of Malabar on the other. This knowledge encompassed a very effective system of cartography using the longitude and latitude. For further details, see Raghava Varier and Rajan Gurukkal, *A Primary Report on the History of Lakshadweep Islands*, Unpublished Project document, 2001, p.7.

[22] *Drkarma lambanabhyam vinastiti kalpitatu vihaganam Swapne drstam dhanamiava bhave thi tham Chintya meva he ganakah*

[23] N.V.P. Unithiri, "Astronomy and Mathematics in Medieval Kerala with special reference to the Nila Valley", N.V.P. Unithiri (ed.), *Indian Scientific Traditions*, University of Calicut, 2003, pp.189-196.

[24] The essence of *Drkganita* is as follows: The exact position of a planet arrived at by the calculation through the *parahita* computation is incorrect since no account is taken of the observed position at different times of the year. The necessary corrections are then made through *Drkganita*.

[25] See K.V. Sarma, "Anpathianchu Kollathe Thapassu", *Mathrubhumi Weekly*, Kozhikode, 7.10, 1956, p. 29.

[26] *Ibid.*, p.3.

[27] Uranad Kunhan Pillai (ed.), *Aryabhadeeyam*, Trivandrum Sanskrit Series No. 185, Thiruvananthapuram, 1957, p. 15.

PART – II

Work of the Kerala School of Mathematics and Astronomy : Some Exemplars

4

KERALA MATHEMATICS - Motivation, Rationale and Method

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Introduction

In 1637, René Descartes wrote in *La Geometrie*: "The ratios between straight and curved lines are not known, and I believe cannot be discovered by human minds, and therefore no conclusion based upon such ratios can be accepted as rigorous and exact." (p. 91).[1] In the earliest written record containing Indian mathematics, the *Śulbasūtra*s (c. 500 – 800 BC), there are instructions for estimating the length of a curved line by laying a rope along the curve and then measuring the length of the rope. A thousand years later, in the *Āryabhaṭīya,* there are two verses on the subject. Verse 10 provides an accurate measure of the curved circumference of a circle in terms of the diameter (a straight line)[1]. Verse 11 indicates somewhat cryptically how an Indian sine (Rsine) table may be derived by approximating small arcs by line segments.[2] This idea of the existence of a mathematical relationship between a curved and straight line would continue to inspire the work of the Kerala School of

mathematics and astronomers. And it would determine the two directions in which Āryabhaṭa would continue during the next thousand years to influence developments in Indian mathematics: work on infinite series relating to circular and trigonometric functions and the construction of accurate sine and cosine tables.

The primary mathematical motivation for the Kerala work on infinite series was the recognition of the impossibility of arriving at an exact value for the circumference of a circle for a given diameter (i.e., or the problem of the incommensurability of π).[3] Nīlakaṇṭha (1444- 1545) explained this conundrum in his *Āryabhaṭīyabhasya*[4] in the following terms:

> "Why is only the approximate value (of circumference) given here? Let me explain. Because the real value cannot be obtained. If the diameter can be measured without a remainder, the circumference measured by the same unit (of measurement) will leave a remainder. Similarly, the unit which measures the circumference without a remainder will leave a remainder when used for measuring the diameter. Hence, the two measured by the same unit will never be without a remainder (*alpāvayavatvam*). Though we try very hard we can reduce the remainder to a small quantity but never achieve the state of 'remainder-less-ness' (*niravayavatvam*). This is the problem."

This clarification was prompted by the following Verse 10 of the *Āryabhaṭīya*[5]

> "Add 4 to 100, multiply by 8, and add 62,000. The result is approximately the circumference of a circle whose diameter is 20,000."

The word "approximately"[6] gave food for thought. Faced by this problem, Śaṅkara Vāriyar and Nārāyaṇa proposed the following in the *Kriyākramakarī*[7]:

> "Even by computing the results progressively (and indefinitely), it is impossible theoretically to come to a final value. So, one has to stop computation at that stage of accuracy that one wants and take the final result arrived at ignoring previous results."

To understand the broader rationale of what is being suggested here, it is necessary to recognize that there are two main approaches to calculating the circumference. The first is an algebraic recursion relation - involving a square root - that converges to an exact value. In modern notation, this relation may be expressed as:[8]

$$x_2 = 1, \quad x_{n+1} = \frac{\sqrt{1 + x_n^{\,2}} - 1}{x_n}, \qquad \pi = \lim_{n\to\infty} 2^n x_n$$

The second method starts as a way of avoiding square roots in the calculation of the circumference by obtaining a finite series which converges to the circumference as the number of terms grows. This approach requires finding the length of an arc by approximating it to a straight line. Known as the method of direct rectification, it involves summation of very small arc segments and reducing the resulting sum to an integral.

The second approach is based on an interesting geometric technique. The tangent is divided up into equal segments while at the same time forcing a sub-division of the arc into **unequal** parts. This is required since the method involves the summation of a large number of very small arc segments, traditionally achieved in European and Arab mathematics by the 'method of exhaustion'[9], where there was a sub-division of an arc into **equal** parts. The choice by the Kerala School of this "infinite series" technique rather than the "method of exhaustion" was not through ignorance of the latter. But, as Jyeṣṭhadeva implies in the *Yuktibhāṣā*, this approach avoids tedious and time-consuming root-extractions.[10]

There are certain aspects of the "tool kit" used by the Kerala mathematicians that will be elaborated in the next section. Briefly, the derivation of infinite series of both circular and trigonometric functions deployed two results from elementary mathematics that have a long history in India:

(i) The Pythagorean theorem which dates back to the *Śulbasūtras*; and
(ii) The properties of similar triangles which are little more than a geometrical version of the "rule of three" *(trairāśika)* of which the earliest treatment has been traced to the *Śulbasūtras*.[11] However, it would seem that Verse 26[1] of the *gaṇita* section of *Āryabhaṭīya* was a more likely source for the Kerala mathematicians.

The Kerala derivation of the infinite series for circumference deploys an ingenious iterative re-substitution procedure[13] to obtain the binomial expansion for the expression $\frac{1}{1+x}$ and then after a number of repeated summations (*varamsamkalita*-s) of series, arrive at what must be the most remarkable part of the derivation, an intuitive leap that leads to the following asymptotic formula:[14]

$$\lim_{n \to \infty} \frac{1}{n^{p+1}} \sum_{j=1}^{n} j^{p} = \frac{1}{p+1}$$

It was soon realised that in the case of the infinite series expansions for circumference (C) in terms of diameter (d) attributed to Madhava:

$$C = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \ldots\ldots \qquad \text{(I)}$$

was not particularly useful for making accurate estimates of the circumference for a given diameter (i.e., estimating π) because of the slowness of the convergence of the series. To stop the computation of the C in (I) at any desired stage, as the quotation from *Kriyākramakarī* recommends, means that the infinite series has to be truncated and such truncation results in some error in the value of C obtained. The objective was to minimise this truncation error (or to compensate for the loss of terms due to truncation). This gave impetus to developments in two directions: (a) rational approximations by applying corrections to partial (or truncated) sums of the series; (b) obtaining more rapidly converging series by transforming the original series.

There was considerable work in both directions which are discussed in detail in both the *Yuktibhāṣā* and the *Kriyākramakarī*. What the work exhibits is a measure of understanding of the concept of convergence, of the notion of rapidity of convergence and an awareness that convergence can be speeded up by transformations.

As an illustration of the remarkable efficiency of some of the corrections introduced, consider the following examples. In his commentary on the *Āryabhaṭīya*, Nīlakaṇṭha referred to an accurate value of Madhava's: "It is stated by the learned that the circumference of a circle with diameter 900 000 000 000 is 2 827 433 388 233." This gives $\pi = 3.14159265359$ which is correct up to 11 decimal places.[15] In the *Kriyākramakarī*, there are other approximations given. According to one statement the circumference is 355 for a diameter measure 113. Another estimate gives an implicit value for π as 3.141592654 (C = 104348 and d = 33215). Consider yet another example from the *Yuktibhāṣā*. What is required is to evaluate the circumference of a circle with a diameter of 10^{11}. Without the correction and using Series (I) given above with the number of terms on the right-hand side as nineteen, the circumference is about 3.194×10^{11}. However, incorporating one of the corrections[16] gives the circumference as $3.1415926529 \times 10^{11}$ which is correct to 9 places. And the interest in increasing the accuracy of the estimate seemed to have continued for a long time, so that as late as the nineteenth century, Śaṅkara Varma, the author of *Sadratnamālā*, estimated the circumference of a circle corresponding to a diameter measure of 1 *parārdha* (10^{17}) as 314,159,265,358,979,324 correct to 17 places[17].

Deriving the Infinite Series for the Circumference: A Study of Method

The Tools

The derivation of (I) above by the Kerala mathematicians required a 'toolkit' containing four main results listed below. The first two have already been referred and are sufficiently

well-known to require any further elaboration. The last two will be discussed here, with the source book being *Yuktibhāṣā*. Modern notation will be deployed for ease of exposition.

(i) Properties of similar triangles
(ii) The Pythagorean Theorem
(iii) Summation of a geometric series
(iv) Establishing the behaviour of a certain quotient taken to its limit

(iii) Summation of a Geometric Series

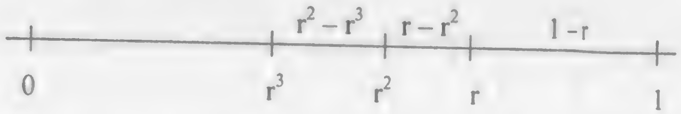
Let r be a number such that $-1 < r < 1$. Show that

$$1 + r + r^2 + r^3 + \ldots\ldots\ldots\ldots = \frac{1}{1-r} \qquad \text{(Infinite Series)}$$

$$1 + r + r^2 + r^3 + \ldots\ldots\ldots + r^k = \frac{1-r^{k+1}}{1-r} \qquad \text{(Finite Series)}$$

A Visual Demonstration

For r <1 only



$$(1-r) + (r - r^2) + (r^2 - r^3) + \ldots\ldots\ldots\ldots = 1$$

Or

$$(1-r)(1 + r + r^2 + r^3 + \ldots\ldots\ldots\ldots) = 1$$

Therefore

$$1 + r + r^2 + r^3 + \ldots\ldots\ldots\ldots = \frac{1}{1-r}$$

A Geometric Demonstration

In Figure I below, triangles ABC and DEC are similar. So

$$\frac{AB}{DE} = \frac{AC}{DC} \qquad (1)$$

But AC=1, DE = 1 and DC = 1- r

Therefore (1) reduces to

$$AB = \frac{1}{DC} = \frac{1}{1-r}$$

But $AB = 1 + r + r^2 + r^3 + \ldots\ldots\ldots\ldots$

So $$1 + r + r^2 + r^3 + \ldots\ldots\ldots\ldots = \frac{1}{1-r} \qquad (2)$$

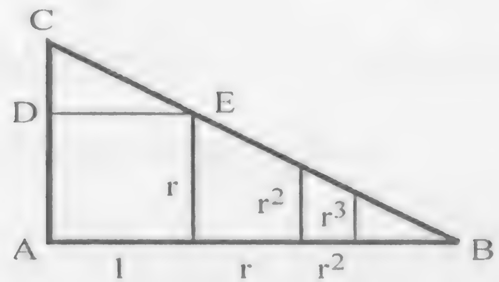


Figure I

In the finite case,

$$(1-r) + (r - r^2) + (r^2 - r^3) + \ldots\ldots\ldots + (r^k - r^{k+1}) = 1 - r^{k+1}$$

Factor out (1- r) from all terms on LHS and rearrange to get:

$$1 + r + r^2 + r^3 + \ldots\ldots\ldots + r^k = \frac{1-r^{k+1}}{1-r} \qquad (3)$$

(iv) Behaviour of a Quotient taken to its limit

In mathematics, this quotient arose from two sets of problems:

(a) Finding the area under a curve.
(b) Construction of a tangent to a curve

(a) Area under the curve $y = x^2$ (For r < 1)

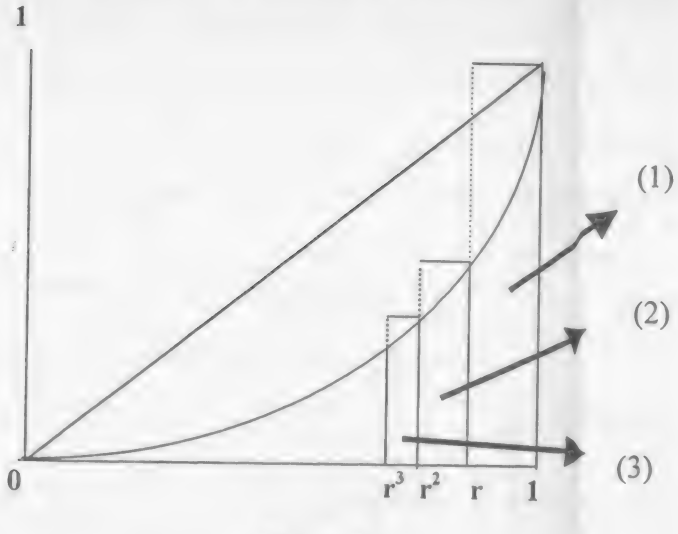


Figure II

Area of (1) = 1 x (1 - r) = (1 – r)
Area of (2) = $r^2 (r - r^2) = r^3 (1 - r)$
Area of (3) = $r^4 (r^2 - r^3) = r^6 (1 - r)$

So the area of the 'endless' staircase generated by increasing powers of **r**, shown in Figure II, is:

$(1 - r) + r^3 (1- r) + r^6 (1-r) + \ldots\ldots\ldots\ldots$

$= (1 - r) (1 + r^3 + r^6 + \ldots\ldots\ldots)$

$= (1 - r) . \dfrac{1}{1-r^3} \quad = \quad \dfrac{1-r}{1-r^3}$

Or more generally for the curve $y = x^{k-1}$ the area is

$= (1 - r) . \dfrac{1}{1-r^k} \quad = \quad \dfrac{1-r}{1-r^k}$ (4)

Question: What happens to $\dfrac{1-r}{1-r^k}$ as r gets nearer to 1 and k increases?

If k = 2, $\dfrac{1-r}{1-r^2} = (1 + r + r^2)^{-1} = 1/3$ as $r \Rightarrow 1$

If k = 3, $\dfrac{1-r}{1-r^3} = (1 + r + r^2 + r^3)^{-1} = 1/4$ as $r \Rightarrow 1$

If k = 4, $\dfrac{1-r}{1-r^k} = (1 + r + r^2 + r^3 + r^4)^{-1} = 1/5$ as $r \Rightarrow 1$

Or more generally,

$$\frac{1-r}{1-r^k} = (1 + r + r^2 + r^3 + r^4 + \ldots . r^k)^{-1} = \frac{1}{k+1} \text{ as } r \Rightarrow 1 \qquad (5)$$

(b) Construction of a Tangent to a Curve

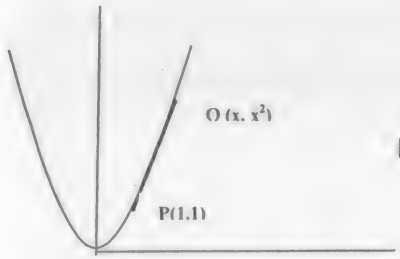


Figure III(a)

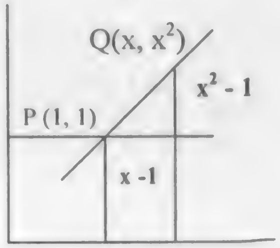


Figure III(b)

$\tan\theta = \dfrac{Rise}{Run} = \dfrac{x^2-1}{x-1}$

The slope of the tangent line to the curve $y = x^2$ at P is:

$$\frac{x^2-1}{x-1} = \frac{-(1-x^2)}{-(1-x)} = \frac{1-x^2}{1-x} = 1 + x = 2 \quad \text{as } x \to 1 \quad (1)$$

It is easily shown that for the curve $y = x^3$, the slope of the tangent to that curve at P is:

$1 + x + x^2 = 3$ as $x \to 1$

Or more generally, for the curve $y = x^k$, the slope of the tangent to that curve at P is:

$1 + x + x^2 + x^3 + x^4 + \ldots x^k = k+1$ as $x \to 1$ (2)

And finally,

(c) <u>A Problem in '*Sunyaganita*' (Zero Arithmetic</u>): What is Zero over Zero?

This may be expressed in modern mathematics as:

Question: Evaluate $\dfrac{x^2-1}{x-1}$ **as** $x \to 1$ (1)

Note that when $x = 1$, (1) becomes 0/0 which is clearly not possible since division is a form of inverse multiplication so that

$\dfrac{0}{0} = 0 \times ? = 0$ where ? can be any number except 0

So what is the solution to (1) ?
Recall that:

$$\frac{1-x^{k+1}}{1-x} = 1 + x + x^2 + \ldots\ldots\ldots\ldots\ldots + x^k$$

For $k = 1$ and where $x \to 1$,

$$\frac{1-x^2}{1-x} = 1 + x = 2$$

For $k = 2$ and where $x \to 1$,

$$\frac{1-x^3}{1-x} = 1 + x + x^2 = 3$$

For the general case k where $x \to 1$,

$$1 + x + \ldots\ldots + x^k = k + 1$$

<u>Derivation of the Arctan and π Series: The Modern Approach</u>

Recall the infinite series given earlier

$$C = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \ldots \text{ (I)}$$

Since $\pi = C/d$, (I) may be rewritten as:

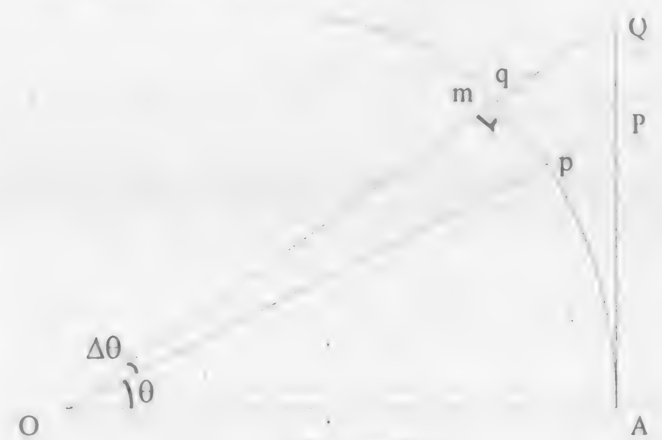


Figure IV

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\ldots \quad \text{(Ib)}$$

The derivation of (Ib) today is a simple affair today using calculus and involves five steps. In Figure IV. Apq is a section of a circle of unit radius OA

Step 1: $\partial\theta = \text{arc pq} \approx \text{pm} = \dfrac{\partial(\tan\theta)}{(1+\tan^2\theta)}$ where OA = Op = 1

Step 2:

So $\theta = \int \dfrac{dt}{1+t^2} = \int(1 - t^2 + t^4 - ....)dt = t - \dfrac{t^3}{3} + \dfrac{t^5}{5} - \dfrac{t^7}{7}......$

where t = tanθ

Step 3 It is next shown that the last term of this series in Step 2 is:

$$(-1)^{n+1}.\frac{t^{2n+2}}{(1+r^2)}dt \Rightarrow \text{O as } n \to \infty \text{ if } |t| \le 1$$

Step 4: Or, arctanθ can be represented as an infinite series of the form:

$$\arctan\theta = \tan\theta - \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} - \frac{\tan^7\theta}{7}.... \text{ if } |\tan\theta| \le 1 \ (*)$$

Step 5: For tanθ = 1 or θ = 45^0 = π/4 radians, the above series (*) becomes

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}.......... \quad ..... \qquad \text{(Ib)}$$

Or equivalently

$$C = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + ... \qquad \text{(Ia)}$$

(*) was first investigated in Europe by the Scottish mathematician, James Gregory, in 1671. Two years later, the German philosopher and mathematician, Gottfried Wilhelm Liebniz, discovered (Ib) using a different approach to that of Gregory.[18] But nearly three centuries earlier, both series were known in Kerala.

Derivation of the Arctan and π Series: The Kerala Derivation

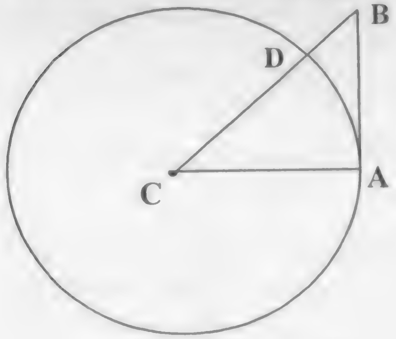


Figure V (a)

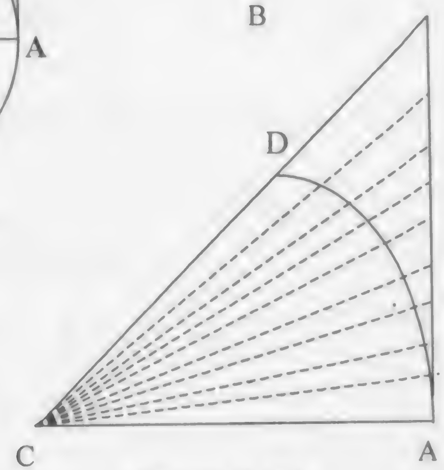


Figure V (b)

Figure V (a) represents a circle of unit radius with the angle at the centre of the circle equalling 45 degrees. AB is the tangent at point A. It would follow that since AB = 1, circumference (C) =2π and arc AD = ⅛C = π/4. Figure V(b) represents the line segment AB being divided into a number of sections (say *n*) of equal length. Note that the length of each section is 1/n Note also that at the same time arc AD cut into n unequal sections. The objective is to estimate the lengths of each of these small arcs and sum these estimates to obtain in turn an estimate of the length of arc AD which we know is equal to π/4 Without loss of generality, consider the special case of n = 5, as shown in Figure V(c). We wish to estimate the portion of the arc labelled EF that corresponds to the line segment 3/5 to 4/5, labelled GH. The objective is to estimate IF (Figure V(d)) which will be a good approximation for the arc EF when GH is very small.

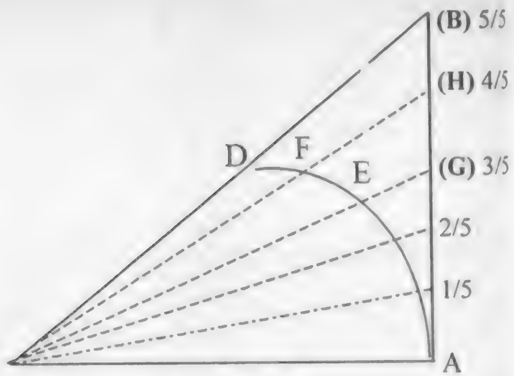


Figure V(c)

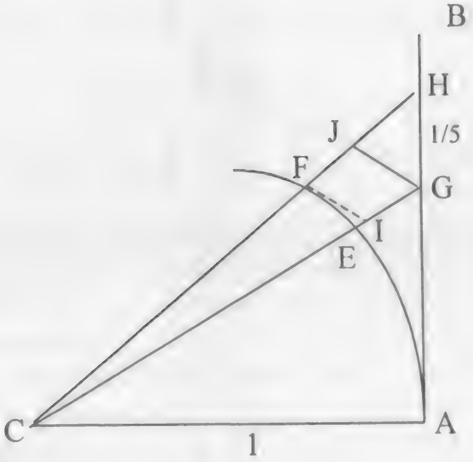


Figure V(d)

Proof: In Figure V(d) Right-angled triangles GJH and CAH are similar.

Since AC = 1 and GH = $\frac{1}{5}$, it would follow that

$$GJ = \frac{1/5}{CH} \qquad (2)$$

Also since right-angled triangles CFI and CJG are similar, it would follow that

$$\frac{FI}{GJ} = \frac{CF}{CH} \Rightarrow FI = \frac{GJ.CF}{CJ}$$

But CF = 1. So $FI = \frac{GJ}{CJ}$ (3)

When n is large, GH is small and so is JH. Therefore CH is a good approximation to CJ

Or $$\frac{CH}{CJ} \approx 1$$

Using this fact, rewrite (3) as

$$FI = \frac{GJ}{CJ}.\frac{CH}{CH}$$

So

$$FI = \frac{GJ}{CH} \qquad (4)$$

Combining (4) with the earlier result given in (1), we get

$$FI = \frac{1/5 \big/ CH}{CH} = \frac{1/5}{CH^2} \qquad (5)$$

Now, note that CH is the hypotenuse of right-angled Triangle CAH. So

$$CH^2 = CA^2 + AH^2 = 1 + (4/5)^2 \qquad (6)$$

Combining (5) and (6) we get

$$FI = \frac{1/5}{1+(4/5)^2}$$

So an estimate of arc FE is:

$$\text{arc FE} \approx FI = \frac{1/5}{1+(4/5)^2} \qquad (7)$$

In the general case, the numerator (1/5) will be replaced by (1/n) and the (4/5) in the denominator of (7) will be replaced by a fraction that describes the length AH. In the present case of n =5,

the length of the arc AD is the sum of the estimates of five short arcs where each estimate will correspond to that calculated for arc FE in (7). In each case the numerator is 1/5 and the denominator is different, each going from $1 + (1/5)^2$ to $1 + (5/5)^2$.

The total estimate of arc length AD will be:

$$\frac{1/5}{1+(1/5)^2}+\frac{1/5}{1+(2/5)^2}+\frac{1/5}{1+(3/5)^2}+\frac{1/5}{1+(4/5)^2}+\frac{1/5}{1+(5/5)^2} \qquad (8)$$

What we are really interested is what happens to the summands as n increases from n = 5 to larger numbers. (8) can then be generalised to:

$$\frac{1/n}{1+(1/n)^2}+\frac{1/n}{1+(2/n)^2}+\frac{1/n}{1+(3/n)^2}+\cdots\frac{1/n}{1+(n/n-1)^2}+\frac{1/n}{1+(n/n)^2} \qquad (9)$$

Factoring out the numerator 1/n in (9) gives:

$$\frac{1}{n}\left[\frac{1}{1+(1/n)^2}+\frac{1}{1+(2/n)^2}+\frac{1}{1+(3/n)^2}+\ldots\ldots+\frac{1}{1+(n-1/n)^2}+\frac{1}{1+(n/n)^2}\right] \qquad (10)$$

Recall the earlier result for the summation of a geometric series:

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \ldots\ldots\ldots$$

Replace r by – s where 1> s > 0

$$\frac{1}{1+s} = 1 - s + s^2 - s^3 + \ldots\ldots\ldots \qquad (11)$$

Applying (11) to each term in the bracket in (10)

$$\frac{1}{1+\left(\frac{1}{n}\right)^2} = 1 - \left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)^4 - \left(\frac{1}{n}\right)^6 + \ldots\ldots\ldots$$

$$\frac{1}{1+\left(\frac{2}{n}\right)^2} = 1 - \left(\frac{2}{n}\right)^2 + \left(\frac{2}{n}\right)^4 - \left(\frac{2}{n}\right)^6 + \ldots\ldots\ldots$$

$$\frac{1}{1+\left(\frac{3}{n}\right)^2} = 1 - \left(\frac{3}{n}\right)^2 + \left(\frac{3}{n}\right)^4 - \left(\frac{3}{n}\right)^6 + \ldots\ldots\ldots$$

$$\frac{1}{1+\left(\frac{n-1}{n}\right)^2} = 1 - \left(\frac{n-1}{n}\right)^2 + \left(\frac{n-1}{n}\right)^4 - \left(\frac{n-1}{n}\right)^6 + \ldots\ldots$$

The last term in the brackets in (9) reduces to

$$\frac{1}{n}\left[\frac{1}{1+\left(\frac{n}{n}\right)^2}\right] = \frac{1}{2n}$$ which tends to 0 as n increases.

Adding the above set of equations column wise:
The first column sum of *(n - 1)* 1's gives n – 1.
The second column sum, after factoring out the denominator n^2, gives:

$$-\frac{1}{n^2}\left[1^2 + 2^2 + 3^2 + \ldots\ldots + (n-1)^2\right]$$

Similarly, the subsequent column sums after factoring out of the denominator n^2 can be written as:

$$\frac{1}{n^4}\left[1^4 + 2^4 + 3^4 + \ldots\ldots + (n-1)^4\right]$$

$$-\frac{1}{n^6}\left[1^6 + 2^6 + 3^6 + \ldots\ldots + (n-1)^6\right]$$

....

Substituting the column sums given above in (10) gives:

$$\frac{n-1}{n}-\frac{1}{n^3}\left[1^2+2^2+3^2+...+(n-1)^2\right]+$$
$$\frac{1}{n^5}\left[1^4+2^4+3^4+....+(n-1)^4\right]$$
$$-\frac{1}{n^6}\left[1^6+2^6+3^6+....+(n-1)^6\right]+ .... \quad (12)$$

From an earlier result regarding the behaviour of a quotient taken to the limit, we established that as n approaches infinity:

$$\frac{1}{n^{k+1}}\left[1^k+2^k+3^k+...+(n-1)^k\right]=\frac{1}{k+1} \quad (13)$$

Applying (13) to (12) gives

$$-\frac{1}{n^3}\left[1^2+2^2+3^2+...+(n-1)^2\right]=-\frac{1}{3}$$
$$\frac{1}{n^5}\left[1^4+2^4+3^4+...+(n-1)^4\right]=\frac{1}{5} \quad (14)$$
$$-\frac{1}{n^7}\left[1^6+2^6+3^6+...+(n-1)^6\right]=-\frac{1}{7}$$

....

....

Therefore, the length of arc AD or $\frac{\pi}{4}$ is given by the infinite series:

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+.......... \quad (\text{I b})$$

Or equivalently

$$C=4d-\frac{4d}{3}+\frac{4d}{5}-\frac{4d}{7}+...... \quad (\text{Ia})$$

It is clear that in deriving this series, the Kerala School showed both an awareness of the principle of integration and an intuitive perception of small quantities and operations with such

quantities. However, having come so near to the formulation of the crucial concept of the 'limit' of a function, they shied away from developing the methods and algorithms of calculus, being perfectly content with the geometrical approach which their European counterparts eventually replaced with calculus.[19]

CONSTRUCTION OF THE SINE TABLES: THE KERALA CONTRIBUTION

Over many years, Indian mathematicians have made notable contributions to trigonometry under the caption '*jyotpatti*' (*jyā* + *utpatti* = source of Rsines). This branch of mathematics evolved from astronomical needs, such as computation of latitudes or of position of planets, their movements etc. As Nīlakaṇṭha pointed out in his '*Golasāra*'[20], while explaining the concept of *jyā* (or Rsines), that he was computing sines and cosines because they were required for a discussion of the motion of planets in their respective orbits on the stellar sphere.[21]

As a case study of the advances made in trigonometry by the Kerala School, consider Nīlakaṇṭha's method of computing sine tables as shown in his *Golasāra*[22]. It begins by providing a geometrical method of computing successively the values of Rsines of half angles starting from $30^0 = 1800'$ along the lines indicated in the *Āryabhaṭīya*. The method is then repeatedly applied to find the Rsine of $\left(\frac{1800}{2^n}\right)$ for $n = 1, 2, 3, 4....$ Next, taking the value of Rsine of $\left(\frac{1800}{2^m}\right)$ for some chosen m as the first Rsine value $\left[\text{say } J_1 \left\{= \text{Rsinh for h} = \left(\frac{1800}{2^m}\right)\right\}\right]$, sine table of length $l = 3\times 2^m$ may be constructed.

The geometrical background to this method dates back to the period of Āryabhaṭa I. The trigonometric formula embedded in this description is

$$R\sin\left(\frac{\theta}{2^n}\right)=\frac{1}{2}\sqrt{\left\{R\sin\left(\frac{\theta}{2^{n-1}}\right)\right\}^2+\left\{Rver\sin\left(\frac{\theta}{2^{n-1}}\right)\right\}^2}$$

where

$$Rver\sin\left(\frac{\theta}{2^{n-1}}\right)=1-R\cos\left(\frac{\theta}{2^{n-1}}\right),$$

$$R\cos\left(\frac{\theta}{2^{n-1}}\right)=\sqrt{R^2-\left\{R\sin\left(\frac{\theta}{2^{n-1}}\right)\right\}^2}$$

for $n=1, 2, 3,$.......... which is intended to find the R sine of half angles (in succession by repeated application). Starting from $\theta=30^0=1800'$ for which the value of Rsine is obtained and given as $\frac{R}{2}$, the method is initiated to get Rsine of $900'$ i.e, $R\sin\left(\frac{\theta}{2}\right)$. Proceed similarly with $R\sin\left(\frac{\theta}{2}\right)$ to get $R\sin\left(\frac{\theta}{2^2}\right)$, and so on. If the process is repeated m times $R\sin\left(\frac{\theta}{2^m}\right)=R\sin\left(\frac{1800}{2^m}\right)$ is obtained which is the value of the first R sine in the table of Rsines at intervals of $h=\left(\frac{1800'}{2^m}\right)$. The table will be of length $l=3\times 2^m$.

The procedure for computation of sine and cosine tables of length $l=3*2^m$ at equal arc intervals $h=\frac{1800}{l}$ for m = 0, 1, 2, 3, 4, 5, etc. consists of the following steps.

Step-I: Take $R=\dfrac{21600\times 113}{2\times 355}$ computed using $\pi=\dfrac{355}{113}$ as stated in the rule.

Step-II: Start with $\theta=30^0$, $S_0=RSin30^0=\frac{R}{2}$ and $C_0=RCos30^0=\sqrt{R^2-S_0^{\ 2}}$

Step III: Denote $S_i=RSin\left(\frac{\theta}{2^i}\right)$ and $C_i=RCos\left(\frac{\theta}{2^i}\right)$ and calculate $S_i=\sqrt{S_{i-1}^{\ 2}+\left(R-C_{i-1}\right)^2}$ and $C_i=\sqrt{R^2-S_i^{\ 2}}$, for $i=1, 2, 3,$ ...., m where $m=3, 4, 5,$ .... according to whether the arc interval h is $225'$, $112.5'$, $56.25'$, ...

Step IV: Now for initiating the computation of Rsine and Rcosine tables at interval of $h=\frac{\theta}{2^m}=\frac{30\times 60}{2^m}$, the first tabular Rsine $=J_1$ is the value S_m and the last Rsine is $J_l=RSinlh=R$, the half diameter, where

$$l=\frac{90\times 60}{\left(\frac{\theta}{2^m}\right)}=\frac{90\times 60\times 2^m}{30\times 60}=3\times 2^m, \text{ since } \theta=30^0=30\times 60'$$

Step V: Compute $J_{l-1}=\sqrt{R^2-J_1^{\ 2}}$, $\Delta J_{l-1}=J_l-J_{l-1}$ and $\lambda=2\left(\frac{\Delta J_{l-1}}{R}\right)$

Step VI: Compute $\Delta J_{l-k}=\lambda\times J_{l-(k-1)}+\Delta J_{l-(k-1)}$

Step VII: Now for $k=2, 3, 4,$ ...... $l-2$ compute $J_{l-k}=J_{l-(k-1)}-\Delta J_{l-k}$ and

$$C_{l-k}=\sqrt{R^2-J_{l-k}^{\ 2}}$$

Sine tables of lengths 3, 6, 12, 24, 48, 96, 192, 384,……may be computed using this *Golasāra* algorithm. It is quite interesting to note that Nīlakaṇṭha has referred to the last and the first Rsine differences by the terms *antya* and *ādi khaṇda* without

mentioning that the last Rsine is the 24th. So it may be inferred that Nīlakaṇṭha's rule for determination of the Rsines successively gives a general method for constructing sine and cosine tables. Comparing the values obtained with the corresponding modern values, it can be shown that the *Golasāra* method gives by and large fairly accurate values even up to 20 decimal places computed.[23] Cosine tables may also be constructed similarly.

PROOF AND RATIONALE IN KERALA AND INDIAN MATHEMATICS

It should be noted that Kerala mathematicians not only enunciated their important discoveries or findings in their works but also took care to provide elaborate exposition of their rationale (both within an algebraic and geometrical framework) along with numerical illustrations when necessary. Quite often their discussions started at a basic level and continued to build up the argument in a systematic fashion. This is particularly evident in their expositions of infinite series methods for evaluation of circumference of a circle and in the computation of sine tables. Various advanced concepts such as those of integration as a limiting sum, differentials and infinitesimals, ε_δ techniques of modern analysis, series approximation techniques along with estimation and analysis of error and the like were introduced in a verbal form. The nature of these discussions reveals that a considerable emphasis was placed on the rationale and demonstrations of abstract mathematical concepts for a better understanding of the results discussed therein. Various commentaries elaborating on the somewhat cryptic statements in certain standard treatises such as the *Āryabhaṭīya* and some elaborate commentaries like the *Yuktibhāṣa* and *Gaṇitayuktayaḥ* etc which deal exclusively with detailed exposition of rationales of various important mathematical and astronomical results reflect the great importance laid by the authors to the notion of proof and methodology (*yukti*, *upapatti*) Some of the proofs discussed referred to even ancient works such as the *Śulbasūtras* (800 – 500 BC).

Interesting demonstrations of Baudhāyana's theorem on square of diagonals (better known as the Pythagorean theorem) are given in the *Yuktibhāṣa* and other commentaries: Baudhāyana's demonstrations of the method of evaluation of square root of two are discussed fully by Nīlakaṇṭha in his commentary on the *Āryabhaṭīya*; Detailed demonstrations of various arithmetic and algebraic rules including those for quadrature of a circle by means of infinite series are attributed to Mādhava; Analysis of series and series approximations are given in the *Yuktibhāṣa* and the *Kriyākramakarī*. Geometrical description of method for computation of sines and development of infinite series for sine, cosine and square of sine etc along with their rationales are found in the above-mentioned texts; geometrical validation of various arithmetic and algebraic truths and special geometric treatment of various progressive series are found in the works of Nīlakaṇṭha and Śaṅkara Vāriyar. And visualizing the series and other mathematical concepts are all examples to support our claim that Indian mathematicians in general and Kerala mathematicians in particular paid attention to various kinds of proofs depending on the nature of the results discussed and the learner.

While the analytic rationale helps one to understand and analyze the underlying mathematical concepts, geometrical demonstrations help one to visualize the concepts and derive immediate conviction. For the benefit of those who did not comprehend a geometric demonstration the algebraic one provided would serve and vice versa. In any case, the main purpose of providing different kinds of demonstrations was to remove all doubts and misapprehensions enabling one to reject wrong notions about the concepts under discussion and convince all scholars in the field about the validity and authenticity of their results. The style of rationale and logical presentation of their arguments serve to elevate or enhance and stimulate the intellect of the learners and motivate them to make further study or application of the results. These main intentions behind the exposition of rationale are revealed explicitly by Gaṇeśa in his prefatory note of the commentary *Buddhivilāsini* on the *Līlāvatī* of Bhāskarā II. Gaṇeśa writes: "Whatever that is stated in the *vyakta gaṇita* or *avyakta gaṇita* (arithmetic or algebra) without

rationale (*upapatti*) may not be clearly understood and without misapprehension (*nirbhrānta*) and will not be acceptable to scholars. A rationale (*upapatti*) is perceivable like a mirror and for enhancing the intellect of the learner; I shall describe the rationale (of each) in its fullness".[24]

Now regarding the nature of proofs in Indian mathematics in general and Kerala mathematics in particular[25] we can say that they were descriptive, instructive and argumentative as well as quite flexible depending on their requirement and background of the learner. As such they were informal and not based on any rigid formal deductive system starting from a set of self evident axioms. Pointing out the main difference between Indian proof style and that of their Greek counterparts, Sarasvati Amma wrote: The Indian's aim was not to build up an edifice of geometry on a few self-evident axioms but to convince the intelligent student of the validity of the theorem, so that visual demonstration was quite an accepted form of proof. This leads us to another characteristic of Indian mathematics which makes it differ profoundly from Greek mathematics. Knowledge for its own sake did not appeal to the Indian mind. Every discipline (*Śāstra*) must have a purpose. And since self-realisation and the resulting deliverance from birth and death was the most legitimate purpose of life, those sciences which were supposed to further this purpose directly or indirectly were most assiduously pursued".[26]

In certain unusual cases, the Indian mathematicians were known to have employed some sort of indirect proofs, resembling *reduction ad absurdum* style, for proving certain statements by assuming the corresponding alternate hypothesis and negating it after a deal of logical argumentation based on already known and established facts. To cite an instance of such kind of proof one may refer to Kṛṣṇa Daivajña's commentary *Bijāṅkura* on the *Bījagaṇita* of Bhāskarā II.[27] To show that a negative number cannot have a real square root, Krishna adopts the following style of proof. "A negative number cannot be a square. Then how can it have a square root? One may ask why a negative number cannot be a square. If possible, suppose one says that 'a negative number is a square. Then whose square is it? Definitely not of a positive number because the square of a positive number can always be only positive, being the product of the two equal numbers. Also not of a negative number because here also square being the product of two equals will be a positive number. As such there is no way whatsoever by which we can find a number whose square is negative". Such indirect proof techniques were generally used only for establishing the non existence of certain entities and not for establishing the statements regarding existence. This seems to be another difference between Indian and Greek mathematics.

Indian mathematicians and astronomers also treated direct observation and experimentation (*parīkṣaṇam*) as another kind of proof. Several results that evolved from such scientific observations were accepted and corrected if found necessary from time to time. For instance, the famous *Dṛggaṇita* system of Vaṭaśreṇi Parameśvara is stated to have evolved from fifty five years of his observational and contemplative studies. Thus we can see that, in order to meet their requirements, Indian mathematicians had also adopted the method of practical observations for establishing various truths. They contemplated upon the keen observations made by them and attained various mathematical truths and knew that they have attained the desired result by the power of the human intellect. However to convince others and make them accept and experience the truth of their enunciations and also to enhance their intellect they put forward arguments based on already established facts in a logical manner. Results and theories thus validated were accepted by all and the corresponding enunciations form the *pramāṇas* or established truths for all future reference and use.

As mentioned earlier, while most of the standard mathematical and astronomical treatises in India contain only such established truths or enunciations of various validated results, the commentaries accompanying them carried proofs, derivations and other demonstrations. Such commentarial literature forms a rich source for study of rationale and methodology in Indian mathematics. The absence of proofs and other demonstrations in most of the standard treatises however created a general impression that Indian mathematicians were not at all serious on notion of proof and that they gave scant attention to methodology. Just a cursory search through some of

the elaborate texts and commentarial literature is enough for one to understand that such impression is not at all true. Following the then prevailing oral tradition of imparting knowledge the Indian masters imparted knowledge to their disciples by orally expounding and demonstrating the subject of study and then summed up only the quintessence of their discourses in the form of *sūtras* in verse or prose of great precision. The dearth of writing materials prompted them to minimize the contents of their treatises by jotting down only the results that are essentially needed for future reference. Such precise *sūtras* can be easily recited learnt and recollected. These *sūtras* would be sufficient and significant to those who knew the key to their meaning or to those who had attended the discourses on them, but would appear to be obscure and of little import to those who did not. In order to preserve the elaborate proofs and other demonstrations acquired from their masters some of the disciples who had drawn inspiration from the discourses took up the task of composing elaborate commentaries on works of their choice as accompanying keys for easy understanding of the basic treatise. The commentators endeavoured to compose such keys after making detailed study of the topic, gathering as much information as possible from all available sources. Such accompanying keys which impart life and spirit to the highly precise sūtras mentioned in the basic text are called *paribhāṣās* or *vyākhyas*. They form a perpetual gloss in which the mine of knowledge embedded in the *sūtras* are brought out to light, proved, elaborated or amplified. The commentaries are some sort of independent works in which the enunciations given in the basic treatise are expounded along with detailed proofs, illustrations, derivations and even visual demonstrations whatever necessary to enable the user to understand fully the basic text as well as the subject. Successively quoting at length each verse of the basic treatise the commentators usually proceed to give exposition interpretations and views and establish them from grass root level without leaving room for any sort of misapprehension. The commentators being products of their own times looked upon their past from their own point of view. As such the commentator, who commits himself to several novel ideas, blends his commentary with developments in the field

during his time and impregnates his composition with his own inventions and ideas. This enhances the quality and utility of the commentary. The main aim of the commentators was to update the knowledge in the field and make the study of literature handed down from the former generations easier to the contemporary and future generations. The commentarial literature thus has an important role in disseminating mathematical knowledge acquired by the commentators. Some of the commentaries like the *Kriyākramakarī* and *Yuktidīpika* (of Śaṅkara Vāriyar) on the *Līlāvatī* and *Tantrasaṅgraha* respectively give a set of numerous *saṅgraha ślokas* (summary verses) at the end of each discussion. These *saṅgraha ślokas* not only help the user to recapitulate the quintessence of the detailed discussions given there in but also provide various important enunciations along with their elaborate proofs, derivations and demonstrations in both algebraic and geometric background. Such expository works are thus store house of various kinds of proofs and methodology adopted by Indian mathematicians in formulating various concepts and theories from time to time to satisfy their needs. Kerala mathematical and astronomical literature is rich in such commentaries. In this context, mention may be made of some of the important Indian mathematical and astronomical works that are well known to contain detailed rationale of several mathematical results.

1. Parameśvara's Commentary *Līlāvatī vyākhya* on the *Līlāvatī* of Bhāskarācārya.
2. *Kriyākrmakarī* of Śaṅkara and Nārāyaṇa which is an elaborate commentary on the *Līlāvatī* of Bhāskarācārya.
3. Nīlakaṇṭha's commentary *Āryabhaṭīya bhāṣya* on the *Āryabhaṭīya* of Āryabhaṭa I.
4. *Yuktibhāṣa* of Jyeṣṭhadeva in two parts, the first part dealing with rationale of mathematical enunciations and the second part with astronomical rationale.
5. *Tantrasaṅgraha* of Nīlakaṇṭha Somayāji, with a short commentary Laghuvivṛti, and an elaborate commentary *Yuktidīpika* by Śaṅkara Variyar.

6. *Gaṇitayuktayaḥ* which is a compendium of rationales in Kerala *gaṇita* or astronomy.
7. *Karaṇa Paddhati* of Putumana Somayāji
8. *Sadratnamālā* of Śaṅkara Varma.
9. Govindaswāmi's commentary *Mahābhāskarīya bhāṣya* on *Mahābhāskarīya* of Bhāskara I.
10. The super commentary *Siddhānta dīpika* of Parameśvara which is a commentary on the commentary of Govindaswāmi on the *Mahābhāskarīya.*
11. Gaṇeśa s commentary *Buddhivilāsinī* on the *Līlāvatī* of Bhāskarā II
12. Kṛṣṇa Daivajña's commentary *Bījāṅkura* on the *Bījagaṇita* of Bhāskara II.
13. Bhāskarā II's own commentary *Vāsanābhāṣya* on his own *Siddhānta Śiromaṇi.*
14. *Marīci of Munīśvara* which is a commentary on the *Siddhānta Śiromaṇi* of Bhāskara II.
15. *Siddhānta Sārva Bhauma* of Munīśvara and his own commentary *Āśaya prakāśa* on it.
16. *Siddhānta tatva viveka* of Kamalākara Bhaṭṭa and his own commentary *Śeṣavāsana.*

This is only a very short list from the tip of the large mass of astronomical and mathematical literature that are capable of throwing much light on the understanding of nature of methodology, motivations, notion of proof, nature of mathematical and astronomical contributions etc., if proper attention is paid on study of their contents. Unfortunately, only a small part of the large mass of the literature has been published so far and even from the published materials only a little bit has been just touched upon and studied seriously in depth. Explorative and exhaustive study of the contents of both published and unpublished materials will certainly dispel the general feeling that Indian mathematicians lacked interest in proof and methodology and give us the exact idea of the logic behind their findings and nature of methodology adopted by them to attain the desired results.

CONCLUSION

The major breakthrough in Kerala mathematics was the appearance of mathematical analysis in the form of infinite series and their finite approximations relating to circular and trigonometric functions. The primary motivation for this work was a mixture of intellectual curiosity and a requirement for greater accuracy in astronomical computations. Demonstrations of these results are not completely rigorous by today's standards, but they are nonetheless correct. And these demonstrations may well be chosen for a modern mathematics class room because the approach is more intuitive and therefore more convincing.[28]

A historical and comparative study of infinite service could a suitable vehicle for testing certain perceptions about different mathematical traditions. A widely accepted view among historians of mathematics is that mathematics outside the sphere of Greek influence, such as Indian or Chinese mathematical traditions, was *algebraic* in inclination and *empirical* in practice in marked contrast to Greek mathematics which was *geometric* and *anti-empirical*. Again, many of the commonly available books on history of mathematics declare or imply that Indian mathematics, whatever be its other achievements, did not have any notion of proof. What a comparative study would indicate is the dangers of such categorisations and generalisations. And in a deeper sense it would bring home the point that between different mathematical traditions there are certain basic differences in the cognitive structures of mathematics – differences in their ontological conceptions regarding the existence and nature of mathematical objects and methodological conceptions regarding the nature and ways of establishing mathematical truths.[29]

DISCUSSION

This paper was preceded by an earlier one by the same authors which contained some background to the Indian mathematical tradition, in particular as it developed in Kerala. This was particularly useful, because any study of the emergence of mathematical ideas must be placed into its historical and

intellectual context. It is interesting to note, for example, the importance of astronomical calculation both in India and medieval Europe for religious and social purposes, but in Europe such practices had little influence on later mathematical advances; in Kerala, on the other hand, they led to some sophisticated results on infinite series and circle measurement.

It is also very helpful for the paper to have concentrated on two case studies of the Kerala work: the derivation of the arctan series and the construction of sine tables. In the first case study, the paper identifies three 'tools' known to Kerala mathematicians, apart from the Pythagoras' theorem and elementary results about similar triangles, namely:

(i) the ability to sum an infinite geometric progression;
(ii) an understanding that $(1-x)/(1-x^k) \to 1/(k+1)$ as $x \to 1$;
(iii) a knowledge that Σr^k (for values of r from 1 to n) is approximately $n^{k+1}/(k+1)$.

These facts, in particular (i) and (iii), were precisely those used by early seventeenth-century European mathematicians in their first attempts at quadrature: notably Grégoire de Saint Vincent, Pierre de Fermat, Giles Personne de Roberval, and Bonaventura Cavalieria, all in the period 1620 to 1635. The summation of powers in (iii) is particularly important. We know that Fermat, for instance, probably deduced it from his knowledge of sums of triangular numbers. It would have been useful to know from the paper how the Kerala mathematicians knew or derived this relationship.

The same carefree manipulation of infinite series is as evident in the Kerala derivation as it was later to be in the hands of Wallis and Newton in England (in the period approximately1650 to1665). Only very much later were deeper questions raised and answered about the handling of infinite series, and it is probably true to say that if the objections had seriously been raised earlier, some of the most exciting advances in early modern mathematics would have been delayed or halted. Mathematics, like any creative subject, needs its periods of intuition and thinking out-of-bounds, without too much immediate concern for the details.

Although there are, as there must be, similarities in some of the underlying mathematics of both the Kerala and European derivations, the approaches were very different. In particular, the circle division explained in this paper is unusual and original and would seem to have no counterpart in Europe.

The paper offer us both 'modern' and 'traditional' derivations of the arctan series, and it is interesting to compare the two, but in another sense both are decidedly modern, since both are expressed entirely in modern notation. It would be even more illuminating to have the Kerala derivation transcribed in language closer to that in which it was originally written. It is almost always a useful first step in understanding any historical mathematical text to translate it into modern language and modern notation, but that can only be a first step. The next stage is to return to the language and concepts in which it was originally written, which is the only way we can really begin to enter the mind of the author.

Finally, the discussion of proof in this paper is very welcome, though it is not always clear whether the kinds of justification described by the authors appeared in the original formulations or only in later commentaries? They correctly point out, however, that proof from axioms by deduction is not the only acceptable method of persuasion. In Europe too, much looser arguments were considered acceptable for much of the seventeenth and eighteenth centuries, and were often no better than 'it works, therefore it is correct'. Only in the nineteenth century did an obsession with logically watertight proof again begin to dominate western mathematics.

ENDNOTES

1 See Rene Descartes (1954) *Discourse on Method, Optics, Geometry and Meteorology* translated by Paul J Olscamp, Indianpolis, Bobbs-Merill Co., page 91. It is worth nothing in this context that Descartes refers to a geometric curve that is defined by an equation while the Indian example is a 'mechanical' procedure used in constructing altars.

2 It is interesting to note in this context that the Indians from the very beginning took the attitude that the radial distance should be measured in the same angular units in which the length of the

circumference is measured. This would have been consistent with the modern concept of radians had they not retained the Babylonian sexagesimal division of a circle into 360 parts.

3 "Divide a quadrant of the circumference of a circle (into as many parts as desired). Then from (right-handed) triangles and quadrilaterals, once can find as many Rsines of equal arcs as one likes, for any given radius."

4 Note that approximate formulae are good enough for practical purposes have been known for a long time. Archimedes of Syracuse (287-212 BC) obtained the value $223/71 < \pi < 22/7$ by considering a regular polygon of 96 sides to 2 decimal places. A value for π of 22/7 is perfectly acceptable even today among engineers.

5 See K S Sastry (1930) *Āryabhaṭīya* with *bhasya of Nīlakaṇṭha*, Trivandrum, ii.10, pp. 41-42.

6 See Sastry (1930) Ibid, p. 41

7 Some Western historians of mathematics have argued that the Indians were not aware of the fact that π could never be exactly determined. This confusion may have arisen because of the early mistranslation of the word '*āsannaḥ*' as "approximate" or "rough value" as in the quotation given. The word '*āsannaḥ*' is a more subtle term. It conveys the notion of 'approaching' or 'close to' or 'very near to', all of which reflects in turn a notion of the 'unattainable'. Anything 'unattainable' can never be reached.

8 See KV Sarma (1975) *Līlāvatī of Bhāskarācārya with Kriyākramakarī* of Śaṅkara and Nārāyaṇa, Horshiarpur, p. 377

9 The formula derives from the relation: $\tan\theta = \dfrac{2\tan\theta/2}{1-\tan 2(\theta/2)}$ Now

putting
$\tan\theta = x_n$ and $\tan\theta/2 = x_{n+1}$, we then solve the resulting quadratic equation $x_n x^2_{(n+1)} + 2x_{n+1}x_n - x_n = 0$

10 The basic approach here is to inscribe or circumscribe a regular polygon. The problem then is to find the side of the polygon as a multiple of the diameter. An interesting extension of the method, first suggested by Eudoxus of Cnidus (fl. 375 BC) and then extensively used by Archimedes, was a rigorous alternative to "taking the limit" which the Greeks avoided given their well-known "horror of the infinite". It is based on the simple observation that if a circle is enclosed between two polygons of n sides, then, as n increases, the gap between the circumference of the circle and the perimeters of the inscribed and circumscribed polygons diminishes so that eventually the perimeters of the polygons and the circle would become identical. Or in other words, as n increases, the difference in the area between the polygons and the circle would be gradually exhausted.

11 It could also be argued that the ***unequal subdivision*** method leads to series that ***converge*** but that the ***equal subdivision*** method of exhaustion has no chance of leading to a convergent series or sequence – all that can be achieved are better approximations.

12 For a discussion of the history of the 'Rule of Three', see a recent paper by S.R. Sarma (2002) 'Rule of Three and its Variations in India" in Y. Dold-Samplonius et al (ed) *From China to Paris: 2000 Years of Transmission of Mathematical Ideas*, Steiner, Stuttgart, pp. 133-156

13 This verse may be translated as:
In the rule of three, multiply the *phala* (fruit or p) by the *iccha* (desire or i) and divide by the *pramāṇa* (measure/argument or a).
The result is the fruit of desire, i.e., $f = \dfrac{pi}{a}$

14 A familiar result in elementary algebra relates to the infinite decreasing geometric series $1 - x + x^2 - x^3 + \ldots\ldots\ldots\ldots$ where $x < 1$ is the common ratio with the sum equal to $\dfrac{1}{1-(-x)} = \dfrac{1}{1+x}$

15 This asymptotic relation made its first appearance in Europe in the works of Roberval (1634) and Fermat (1636).

16 K. Samba Siva Sastry (1930) *Āryabhaṭīyam with bhāṣya of Nīlakaṇṭha*, Trivandrum, ii. 10 (p.42)

17 The correction used is to incorporate the following as the last term in (I):
$F(n) = \dfrac{n^2+1}{4n^3+5}$ where n is the number of terms on the right hand side of (I)

18 K.V. Sarma (2001) *Sadratnamāla* of Śaṅkara Varman, INSA, New Delhi, p.26

19 In an attempt to discover an infinite series representation of any given trigonometric function and the relationship between the function and its successive derivatives, Gregory stumbled on the arctan series. He took, in terms of modern notation,

$$d\theta = \frac{d(\tan\theta)}{1+\tan^2\theta}$$

and carried out term by term integration to obtain his result: a procedure not dissimilar to the modern derivation given here. Leibniz's disco ery arose from his application of fresh thinking to an old problem, namely quadrature of a circle. In applying a transformation formula (similar to the present-day rule for integration by parts) to the quadrature of the circle, he discovered the series for π. It must be pointed out, however, that the ideas of calculus such as integration by parts, change of variables and higher derivatives were not completely understood then. They were often dressed up in geometric language with, for example, Leibniz talking about "characteristic triangles" and "transmutation".

20 It is interesting in this context to note about five hundred years after Madhava, Yesudas Ramchandra wrote a book in 1850, entitled *A Treatise on the Problems of Maxima and Minima* in which he claimed that he had developed a new method, consistent with the Indian tradition of mathematics, to solve all problems of maxima and minima by algebra and not calculus. This book was republished in England with the help of the British mathematician, Augustus De Morgan. For further details, see G.G. Joseph (1995) "Cognitive Encounters in India during the Age of Imperialism", *Race and Class*, 36 (3), 1995, pp. 39-56.

21 This is a small text on spherical astronomy summarised in 56 verses in Sanskrit.

22 See *Golasāra Siddhāntadarpaṇam ca* of *Gārgya Kerala Nīlakaṇṭha* Ms No. T 846. B, Transcript copy by Paramesvara Sastry, C.1024.E (K.U.O.R.I and Mss Library, Trivandrum), iii vs.2

23 For details see V. Madhukar Mallayya (2004) "An interesting algorithm for computation of sine tables from the Golasāra of Nīlakaṇṭha", *Ganita Bharati 26,, pp.40 – 55.*

24 As part of the AHRB Project, Dr Mallayya undertook a substantive study of the construction of sine tables from Aryabhata to Nilakantha. In a forthcoming publication, *Golsara's* methodology for constructing Sine Tables will be detailed and used in producing Tables of various lengths. An assessment of the accuracy of these Tables will also be given.

25 V.G. Apte (1937) , *Līlāvatī with Buddhivilāsinī of Gaṇeśa Daivajña and Līlāvatī vivaraṇa of Mahīdhara*, Anandasramom series. No. 107, Poona , (Part I), p.1, vs.4)

26 V. Madhukar Mallayya, "Indian Mathematical Tradition with special reference to Kerala - Methodology and motivations", *An Indian Leap in the Advent of Mathematics*, Delhi (forthcoming)

27 See T.A. Sarasvati Amma (1999) *Geometry in Ancient and Medieval India*, Delhi, p.3

28 Viharilal Vasista (1977) *Bījagaṇita with commentary Bījāṅkura of Kṛṣṇa Daivajña*, Jammu, p.16, vs.13.

29 Geometric intuition and logically deductive reasoning formed the basis of "proofs" both in India and later in Europe. By the end of the nineteenth century, geometry fell out of favour to be replaced by arithmetic and set theory. Thus Bolzano and Dedekind tried to prove that infinite sets exist by arguing that any object of thought can be thought about and thus give rise to a new thought object. Today we reject such proofs and use an axiom of infinity. Starting from a practical orientation and serving practitioners of the astronomical arts, the subject of analysis by its peculiar logic developed eventually into a highly abstract and rarefied entity for the delectations of primarily the professional mathematician.

5

DERIVATION OF THE INFINITE SERIES EXPANSION FOR π AS DEMONSTRATED IN THE *YUKTIBHASA*

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Introduction

The *Yuktibhasa* is a comprehensive text in mathematics and astronomy that contains demonstrations and proofs of the 'Kerala School' from the time of Madhava (fl. AD 1350) The author of the text is Jyeshtadeva who began his studies under Damodara, the son of Parameswara and later continued under Nilakantha. In the *Yuktibhasa,* the author refers to Nilakantha as his teacher. And since we know from different sources, including his own references, that Nilakantha's long life span stretched from 1444 to 1545, we can therefore infer that that Jyeshtadeva lived between 1500 to 1610. Also, Achuta Pisharoti, one of the most prominent disciples of Jyeshtadeva, refers to his teacher in reverential terms in his text *Uparagakriyakramakari*, which was composed in 1592. The location of Jyeshtadeva's birthplace can be easily inferred from the fact that he was locally known as "Parangottu Nambuthiri" which suggests that he was from Triprangodu, near Tirur in Kerala. From other references it is clear that the *Yuktibhasa* was written between 1520 and 1550.

An unusual feature of the *Yuktibhasa* is that it is written in the vernacular prose. The language was a mixture of technical terms from Sanskrit and the rest in local Malayalam. It would appear that the language of the *Yuktibhasa* resembles the language used by the '*vidushaka*' (i.e., clown/commentator) while enacting *Koodiyattam* (i.e., the Sanskrit theatre of Kerala). The reason for deviating in this instance from the long tradition of writing scientific texts in Sanskrit verses may simply be a reflection of the renaissance in vernacular languages throughout Indian subcontinent, which manifested itself in literature and other spheres of life. The sixteenth century Kerala saw a flowering of Malayalam Literature led by Thunjathu Ramanujan Ezhuthachan, the father of Malayalam Literature and a fellow student of Jyeshtadeva, both studying under the same *Guru*, Nilakantha.

This paper is concerned with the mathematics from the *Yuktibhasa* that deals with the derivation of the infinite series expansion for the ratio of the circumference to diameter (or π) of a circle. A literal translation into English of the relevant section from the Yuktibhasa is followed by an interpretation in terms of modern mathematics. This should hopefully provide a lucid picture of the content, style and approach of the original text.

A Literal Translation (with additions and explanations in parenthesis)

"Subsequently, from a desired diameter, without using 'the square-square root method', how to derive the circumference is illustrated (i.e. the procedure for arriving at the circumference of a circle of desired radius without the use of squares and roots)."

"First of all, construct a square, with four sides equal to the given diameter. Then construct a circle, inside the square. The periphery of the circle should touch the mid points of the four sides of the square. Then, through the centre of the circle, construct the East-West Axis line and South-North Axis line, with ends at the meeting point of the periphery of the circle and the midpoints of the sides of the square. Then distance from the East-West axis line to the South-East Corner of the square will

be equal to the radius (half of the diameter). On this (line), put a number of close points, forming equally spaced segments. The more the number of points, the more accurate would be the circumference. Then construct radial lines from the centre of the circle to all these points. The East-West line will be the *base* (adjacent side of a right triangle). The distance on the east arm, between the top end point of each radial line and the East-West axis, will be the *height* (opposite side of right triangle). Thus, for the nearest radial line (i.e., *hypotenuse*) to the East-West axis on the south, one segment will be the *height* (opposite side). For the second radial line (*hypotenuse*), two segments together will form the *height* (opposite side). Thus for other radial lines, segments one by one will be successively added on to form the respective *heights* (opposite sides). The radial line (*hypotenuse)* in the corner of the square (the diagonal of square) will have the maximum *height* (opposite side). Then the *base* or adjacent side for all the radial lines will be the radius, which is the East-West axis. Therefore, the square root of the sum of the square of the radius and corresponding <u>height -square</u> will be equal to each radial line (*hypotenuse*)."

"Now, the first segment, between the E-W axis line and the first hypotenuse, when multiplied by the radius which is also the E-W axis line and is divided by the first radial line, the result will be the perpendicular from the E-W axis line to the first radial line. This perpendicular will be a *base*. The end portion of the next radial line from the meeting point of this *base* will be the '*height*'. The *hypotenuse* will be the segment along the side of the square, between the first radial line and the E-W axis line. This will be a consequent figure (triangle). The antecedent figure (triangle) <u>similar</u> to this, is considered next. The E-W line from the centre of the circle to the centre of the eastern side of the square is *base*. The first radial line is *hypotenuse*. The distance between radial line and the *base* is *height*. To this antecedent figure, the consequent figure is <u>similar</u>. Reason for this is that, opposite side of antecedent triangle is in the <u>same direction</u> (parallel) to the hypotenuse of the consequent triangle or otherwise, the opposite side of the consequent triangle (*height*) is in the same direction (parallel) to the *hypotenuse* of the antecedent figure (triangle). Then the base of the antecedent

figure (triangle), which is E-W axis, is perpendicular to the segment of the square. Here it can also be seen that the *base* of the consequent figure (triangle) that is arrived as the result obtained by using the proportionality relationship, is perpendicular to the *hypotenuse* of the antecedent figure (triangle). These are the reasons for the two figures (first and second triangles) to become similar. Thus, here in two figures, the opposite sides and *hypotenuses* have parallelism and bases or adjacent sides of right triangle, opposite sides of right triangles and hypotenuses, have mutual perpendicularity, resulting in the 'similarity of shapes'. Here if all the three (three sides) have either perpendicularity or parallelism, then also, they become similar. This (similarity) could be understood by an analogy with an inclined rafter of a square shaped pavilion (with pyramidal roof system) which is a hypotenuse of antecedent figure (triangle) with opposite side as the *eave line*. This is similar to the consequent figure (triangle) where the slot length provided (in the rafter) for the 'wooden pin' is hypotenuse of the consequent figure (triangle). The opposite side to this hypotenuse, is the offset of the inclination of slot along the side of the rafter. And since the opposite side (in antecedent triangle) and *hypotenuse* (consequent triangle) are parallel to each other, the slope of the slot for the 'wooden pin' is dependent on the slope of the rafter. And by using the 'Rule of Three' (i.e., proportionality relationship), the base of the consequent figure (triangle) could also be arrived at."

"Then, there exists a third triangle. For that triangle the E-W axis is *hypotenuse*. The distance between the E-W axis and the first *hypotenuse*, which is the base or adjacent side of the described consequent figure (triangle), will be the opposite side here. The portion of the first *hypotenuse* (first radial line) between the meeting point of the opposite side and the centre of the circle will be base or adjacent side of right triangle. It is thus."

"Then there is a second antecedent figure (triangle). For that E-W line itself is the *base*. From the end of the base, two segments together, along the side of the square, is *height* or opposite side. The second radial line from the centre of the circle becomes *hypotenuse*. Thus is the second antecedent figure (triangle). Then consider, the consequent figure corresponding to this. The line from the end of point of the first radial line, perpendicular to the second radial line, is the *base*. From the meeting point of this base to the end of the radial line is the *height*. The second segment in the side of the square is the hypotenuse. This is the second consequent figure (triangle). When the second rafter becomes the antecedent hypotenuse, the two spans of the rafters together becomes the antecedent *height* (opposite side of the right triangle). Hence the length of the second rafter will have more length than the first rafter. Proportionately the length of the slot for the *wooden pin* also will be more. That (the length of the slot) will be the hypotenuse of consequent figure (triangle) and will be parallel to the *eave line*, which is the opposite side of the antecedent figure (triangle). How the inclination of the rafter and the inclination of the slot are varying proportionally, in the same way the antecedent figures (triangle) and consequent figures (triangle) considered here are also correlated. Here at the end the E-W line, the second segment in the side of the square is multiplied by the radius, which is the base of the antecedent figure (triangle) and divided by the second radial line (which is the fourth proportion). The result is the base of the consequent figure (Right triangle). Assuming this base as opposite side (height) and then portion of the second radial line between the meeting point of this height and the centre of the circle as adjacent side (base) and the first radial line as *hypotenuse* there exists a third triangle, here also."

"Thus, there are only three triangles in each of the segmental portions starting from the E-W line to the diagonal of the square. There, when each segment from the E-W line (length) up to the corner of the square, when is multiplied by the E-W line (radius) and divided by the longer radial line meeting that segment, the result will be the perpendicular distance to the longer radial line from the preceding radial line. This will be the adjacent side (*base*) of the consequent figure. The same will be again opposite sides (*heights*). The portion of the radial line from the meeting point of this *height* to the centre of the circle will be the *base*. Then the shorter of two radial lines touching (the ends) each segment will be *hypotenuse*. Thus exist certain triangles, which are similar to certain other triangles. These are antecedent figures

and consequent Figures. The antecedent figures (triangles) are those ones constructed with in the circle. Here the *hypotenuses* of the antecedent triangles are equivalent to radius and this will be a third proportion. The perpendicular distance from the end of this radius is fourth proportion. Thus, this will be the Half Chord (Sine Chord) along the circumference of the circle in that portion between the radial lines. Thus the result obtained by multiplying the length of segment from E-W line twice with the radius(square of the radius) and divided by the product of the radial lines (lengths) associated with each segment, will be the Half Chord (Sine Chord) along the circumference between the radial lines. Here, if the segments along the side of the square are infinitesimally small, the *sine chords* will be more or less (equal to the) *arc lengths*."

"Here, since the sides of the squares are equally divided multiplicands are same and the multipliers are also equal as square of the radius. The divisor, being the product of the radial lines, on lower and upper ends each segment, they are different. But in this case, the product of the radial lines (distances) can be assumed as the half of the sum of the squares of the radial lines (lengths), because of the fact that results are numerically comparable. When it is like that, dividend may be divided by the square of radial lengths separately and added together and then made half. To this, the result obtainable by the division of the half sum of the squares of radial lengths will match."

"There, we shall consider, the division of the segments with the squares of the northern radial lines starting from the E-W axis. There the first one will be E-W axis line itself. When, it is divided, by the square of the E-W line (radius), the divisor and multiplier are same, and hence the result will be equal to segment it self. Then consider the last radial line, which is diagonal line. When divided by its square, the result will be a half of the segment, because the square of the last radial line is two times the square of the radius. When the double of the multiplier is divisor, the half of the multiplier will be result. Here, there are two radial lines touching the first and the second end points of each segment. Of these, the difference obtained from the sum of the results got by dividing the squares of the first radial lines and the results obtained from the sum of the

results got by dividing the squares of the second radial lines will be the difference between the first term of the first set and the last term of the second set. That will be half of a segment. The other terms in both sets will be same since the divisors are same. From second term to the last but one, the values are identical. It (difference) will be half of a segment, since the result obtained by the first divisor is equal to one segment and last divisor is equal to half a segment. When half the sum of the squares of the radial distances is considered, the difference is only one-fourth of a segment. As the segments are very small, we can ignore this *one-fourth of one segment*. Thus the divisors can be taken as squares of either one of the radial lines. There, (out of two radial lines), associated with each segment, the longer radial line shall be considered. Then multiply each segment by squares of radius and divide by squares of longer radial line. The result will be the *Sine Chords* along the circumference, between the radial lines."

"Here, difference got by deducting the result obtained by multiplying each segment by *"Gunaharantharam"* then divided by the square of the length of the radial line, from each segment, will be same as the Sine Chords along the circumference. Here the *"Gunaharantharam"* will be the square of the sum of the segments from the E-W line to each of the radial line under consideration. The square of the Radius will be the multiplier. There, if it is multiplied by the "*Gunaharantharam*" and divided by Multiplier, and if the Multiplier is less than the divisor, there will be many (infinite) resulting terms. There if we subtract from one term, the result obtained by multiplying that term with *Gunaharantharam* and divided by Multiplier, the difference will be same as original. There, in the rectification (correction) term also, if we multiply by the *Gunaharantharam* and divide by Multiplier, there also we have to subtract something as in the previous case. There in the case of second correction term also, if it is multiplied by the *Gunaharantharam* divided by the divisor, as a correction to this rectification term there will be third term. Here also if we divide it by the Multiplier there will be a fourth correction term. Thus if we divide all terms by Multiplier, the rectification series will never end, till we divide it by the divisor. If we are not dividing by the divisor at all, the

resultant series will be never ending (infinite). However when terms become infinitesimally small, it could be ignored."

"If proceeded like this, first term will be the sum of the multiplicands. This will be the sum of the segments along the side of the square and is equal to radius. Secondly, the result that is to be subtracted from this. The third one is that to be subtracted from the second. Thus all the odd ones shall be added together and then the sum of the even ones shall be deducted and the balance will be the one-eighth of the circumference, since the multiplier is smaller (lesser). In the case where Multiplier is greater, all (terms) will only have to be added to the Multiplicand."

"Here since the squares of base and squares of hypotenuse are *Gunaharas*, the *Gunaharantharam* is square of heights. There in the side of the square where that is equally divided, one segment becomes first *height*. Two segments together form second Height (opposite side). Three segments together are third *height* (opposite side). Thus segments, one by one added together will have to be further added up. . They have to be considered as (atomic sized) infinitesimally small, for getting the accuracy of the result. Then consider them sum of squares of full numbers from one, (with one increment) multiplied by the Multiplicand, which is infinitesimally small segment and considered as unity, and divided, by square of radius. Result will be the sum of the first terms. For the second sum, the first sum shall be a Multiplicand and here Multiplicand is comprised of different terms and similarly the *Gunaharantharam* is also different being squares of different corresponding heights. Hence there is no means to multiply with the sum of *Gunaharantharam*. Then the previous Multiplicand which is considered, as unity shall be multiplied twice, by the segmental height will be the *Varga Sankalitham*, which is sum of *Gunaharantharam* and shall be divided by square of radius. The result will be sum of second series."

"Here in the third summation result shall also be got in the same way from the first Multiplicand; the integral (*Sankalitham*) of the square of square (fourth power) will be the multiplicand and square of square of radius (4^{th} power) will be divisor. Here for the integration (*sankalitham*), the variable (*padam*) shall be radius."

"For the next, the *integral of sixth power* will be the numerator and sixth power of radius will be the denominator. Thus further, even powers of radius will be divisors and corresponding integrals will be the multiplier. The integral of square will have three as power, the integral of square of square will have five as power, and the integral of sixth power will have seven as power. Here, the multiplier with three as power (cube) when divided by the divisor with two as power (square), the result will be radius itself. Similarly everywhere, when multiplier is divided by divisor, the result will be radius. Afterwards, since it is required to divide the cube by three, divide radius by three. Then it will be equal to the integral of the square of the radius divided by the square of the radius. Similarly the radius divided by five will be equal to the integral of the fourth power (of radius) divided by the fourth power (of radius). Thus, progressively, the radius divided by the odd numbers, three, five etc (*trisaradi...*) will be terms in the said series. That is told as "*Trisaradi Vishama sankhya bhakthaam swam pradhak-kramat kuryat....*" Here, where the subsequent terms in the series has to the deducted from the preceding terms, there it can be got otherwise by subtracting sum of odd terms from the sum of multiplicands (radius) and then by adding the sum of even terms. That is what is told in '*swam pradhak-kramat kuryat*'."

A Modern Interpretation

In an earlier section of Chapter 6 of the *Yuktibhasa*, a more familiar method of finding the ratio of the circumference and the diameter is described The method involves approximating the circle to a regular polygon having increasing number of sides by an iterative process that involved at each stage finding the length of the sides of a regular polygons of $2n$ sides from a regular polygon of n sides. Thus we could get the length of the side of a regular octagon from a square; and the length of a side of a sixteen-sided regular polygon could be derived from a regular octagon, thirty two-sided polygon from sixteen-sided polygon and so on, such that when n becomes very large, the resulting polygon will approximate a circle. This method is described in the *Yuktibhasa* as the "square–square root method"

because it essentially makes use of the relation between the sides of a right triangle.

More precisely, the relationship $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$ is the essence of this method. The values for π given in Aryabhata's *Aryabhatiya* and Bhaskacharya's *Lilavati* are said to have derived using this method by considering a polygon having n = 256 sides. The value given by *Aryabhatiya* (499 AD) is 62832/20000 (or 3.1416) and that given in *Lilavati* of Bhaskaracharya (1156 AD) is 3927/1250 (or 3.1416). The second ratio is easily calculated as follows. If we keep on iterating from square to octagon, octagon to sixteen-sided regular polygon etc, the side of a 128-sided regular polygon that circumscribes a circle having diameter 1250 units will be 30.6857. Further, the side of a 256-sided regular polygon will be 15.3405. Assuming the circumference of the circle and the perimeter regular polygon having 256 sides are approximately equal, the circumference is the product of 256 and 15.3405, which equals 3927.168. Then the implicit value of π is $\dfrac{3927}{1250} = 3.1416$.

In the present section another method by which the circumference-diameter ratio can be derived is illustrated.

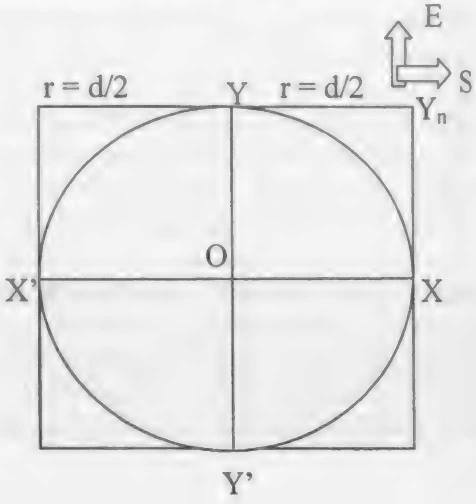

Figure I

Construct a square (whose side is) equal to the diameter and draw a circle as shown in Figure I. Let the diameter be *d*. Let O be the centre of the circle and let *YY'* be the East-West line and XX' be the South-North line. Here the reader may note that in all Indian traditional texts the top of the paper will be the East (In the modern practice, North is shown at the top in all drawings).

Let Y_n be the South-East corner of the square. Then YY_n will be half the diameter or radius (r) of the circle. Create closely located equally spaced points $Y_1, Y_2, Y_3, Y_4, Y_5, \ldots\ldots\, Y_n$ etc. Here it is stated that as the number of points increases, the estimate of the circumference will become more accurate. Thus $YY_1 = \Delta$ where Δ becomes infinitesimally small as $\Delta \to 0$.

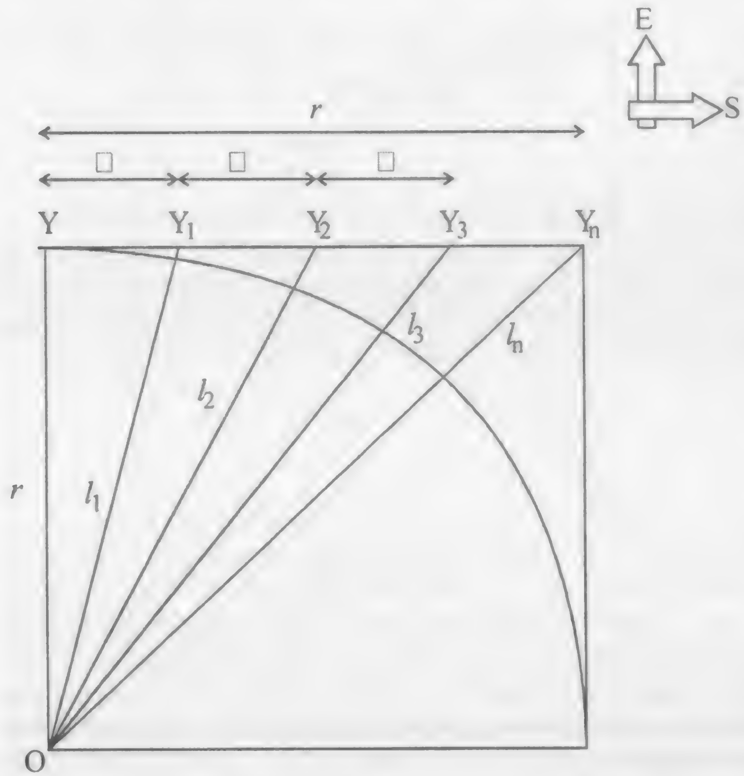

Figure II

Now refer to Figure II. In the right angled triangle, OYY_1.

$$OY_1^2 = OY^2 + YY_1^2$$

Here $OY = r$, $YY_1 = \Delta$ and let $OY_1 = l_1$ and hence $l_1^2 = r^2 + \Delta^2$

Also let $OY_2, OY_3, OY_4, OY_5, \ldots\ldots OY_n$ be $l_2, l_3, l_4, l_5 \ldots\ldots l_n$

Then, $l_2^2 = r^2 + [2\Delta]^2$

$l_3^2 = r^2 + [3\Delta]^2$

$l_4^2 = r^2 + [4\Delta]^2$

....

....

$l_{(n-1)}^2 = r^2 + [(n-1)\Delta]^2$

$l_n^2 = r^2 + [(n)\Delta]^2$, *here* $n.\Delta = r$ *and* $l_n^2 = 2r^2$

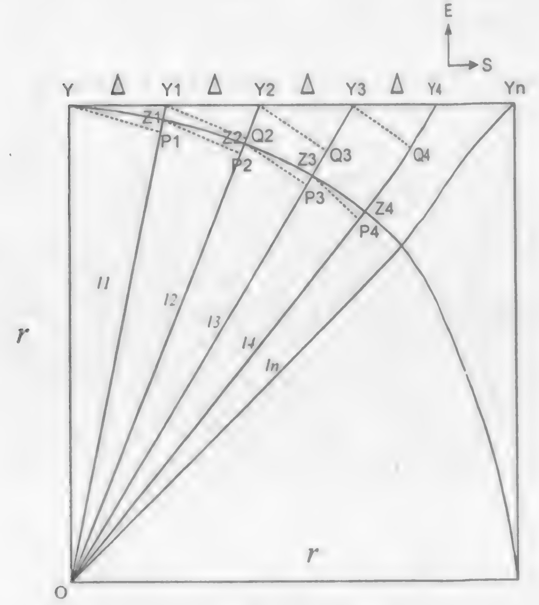

Figure III

In Figure III, Z_1P_2, Z_2P_3 etc are perpendicular drawn from intersection point of circle OY_1 to OY_2, perpendicular drawn from intersection point of circle OY_2 to OY_3 etc.. These distances are half-sine chords (*ardha-jyas*). In other words if $\delta\theta$ is the angle subtended at the center of the circle between radial lines OY and OY_1, OY_1 and OY_2, OY_2 and OY_3 etc.

$$\text{Sin}(\delta\theta) = \frac{YP_1}{r} = \frac{Z_1P_2}{r} = \frac{Z_2P_3}{r} = \ldots\ldots\ldots\ldots = \frac{Z_{(n-1)}P_n}{r}$$

Consider right angled triangles OYY_1 and YP_1Y_1, they are evidently similar (and the reason for the similarity is given in the *Yuktibhasa* and discussed below). Therefore corresponding sides are proportional.

$$\frac{YP_1}{YY_1} = \frac{OY}{OY_1} = \frac{r}{l_1} \quad or \quad YP_1 = \frac{\Delta.r}{l_1}$$

Similarly in the second segment also, right angled triangles $Y_1Q_2Y_2$ and OYY_2 are similar, which yield the relations as given below:

$$\frac{Y_1Q_2}{Y_1Y_2} = \frac{OY}{OY_2} = \frac{r}{l_2}$$

$$Y_1Q_2 = \frac{\Delta.r}{l_2}$$

Here the conditions for two triangles to become similar are also discussed in general. To understand the similarity of triangles, an analogy with rafters and slits (holes) provided in the rafters to receive wooden pins (ties between rafters) is given.

In Kerala of the age of the *Yukthibhasa*, timber constructions in varied forms were common and they formed an important feature of traditional architecture. Hence, the author chose to give an example from building technology. Also, in this part of the world, applications of 'advanced' mathematics were mainly in two areas, astronomy and building science (known as *Vastu Vidya* or *Thachu Sastra*). A contemporary of Jyeshtadeva, Thirumangalath Neelakantan, had written a very popular text on traditional building science, *Manushyalaya Candrika*.

The example given is a square pavilion with a pyramidal shaped roof and a central ridge piece and rafters originating from centre ridge piece in a radial direction set onto the wall plate. These rafters were kept together by another square wooden piece, which was known locally as *vala*. These wooden pins passed through the rafters, and hence there had to be holes (or slots) to receive these pins. The outer-most line of rafters (or the 'eave') was known as *vamata*.

Referring back to Figure III, we can see that there are three triangles between adjacent radial lines, i.e., between l_1 *and* l_2 ; between l_2 *and* l_3;..... etc. In the first segment, i.e. between radius*(r)* and l_1, it may be noted that there are only two triangles, since perpendiculars have merged in that portion. Thus chord length $YP_1 = \frac{\Delta.r}{l_1}$ which can be taken as $\frac{\Delta.r^2}{(r.l_1)}$ by multiplying both numerator and denominator by r. (The same result could be established from properties of similar figures also).

Also, as shown in Figure III, since triangles OY_1Q_2 and OZ_1P_2 are similar,

$$\frac{Z_1P_2}{OZ_1} = \frac{Y_1Q_2}{OY_1} \quad or \quad Z_1P_2 = \frac{OZ_1.Y_1Q_2}{OY_1}. = \frac{r}{l_1}.\frac{\Delta.r}{l_2}$$

$$\because Y_1Q_2 = \frac{\Delta r}{l_2}$$

$$Z_1P_2 = \frac{OZ_1.Y_1Q_2}{OY_1}. = \frac{r}{l_1}.\frac{\Delta.r}{l_2} = \frac{r^2\Delta}{l_1l_2}$$

Similarly, it can be shown that,

$$Z_2P_3 = \frac{r^2.\Delta}{l_2l_3}$$

and also that other half-chord lengths will also be similar. Here these *half-chord lengths* can be taken as the *arc lengths* since the angles subtended by these chords are infinitesimally small.

In the original text, it is said that when *bhuja khanda* (Δ) is '*anu parimana*' or atom sized, the corresponding *half-chord lengths* (*ardhajya* or $r.Sin\delta\theta$), will be arc lengths ('*capa khanda*')

In modern terms this corresponds to

$$Limit \frac{Sin\theta}{\theta} = 1, \; when \; \theta \to 0$$

Thus the total length of the arc ($\frac{1}{8}$ of the circle) will be sum of these chord lengths.

$$\text{Or } \frac{C}{8} = \frac{\Delta.r^2}{rl_1} + \frac{\Delta.r^2}{l_1l_2} + \frac{\Delta.r^2}{l_2l_3} + \ldots\ldots\ldots\ldots + \frac{\Delta.r^2}{l_{(n-1)}l_n}$$

Here the numerators of all terms are the same and are equal to $\Delta.r^2$. Denominators are different for different terms as the product of adjacent radial lines. Now the product of radial lines can be equated to half the sum of their squares using the evident identity

$$(l_1 - l_2)^2 = l_1^2 + l_2^2 - 2l_1l_2$$

so that *when* $(l_2 - l_1) \to 0$

$$l_1l_2 \to \frac{1}{2}\{l_1^2 + l_2^2\} \text{ and } \frac{1}{l_1l_2} = \frac{1}{2}\left\{\frac{1}{l_1^2} + \frac{1}{l_2^2}\right\}$$

If we substitute the above, we have the following

$$\frac{C}{8} = \frac{1}{2}\left\{\frac{\Delta.r^2}{r^2} + \frac{\Delta.r^2}{l_1^2}\right\} + \frac{1}{2}\left\{\frac{\Delta.r^2}{l_1^2} + \frac{\Delta.r^2}{l_2^2}\right\} + \frac{1}{2}\left\{\frac{\Delta.r^2}{l_2^2} + \frac{\Delta.r^2}{l_3^2}\right\} + \ldots\ldots + \frac{1}{2}\left\{\frac{\Delta.r^2}{l_{(n-1)}^2} + \frac{\Delta.r^2}{l_n^2}\right\}$$

Let

$$S_1 = \frac{1}{2}\left\{\frac{\Delta.r^2}{r^2}\right\} + \frac{1}{2}\left\{\frac{\Delta.r^2}{l_1^2}\right\} + \frac{1}{2}\left\{\frac{\Delta.r^2}{l_2^2}\right\} + \ldots\ldots\ldots + \frac{1}{2}\left\{\frac{\Delta.r^2}{l_{(n-1)}^2}\right\}$$

$$S_2 = \frac{1}{2}\left\{\frac{\Delta.r^2}{l_1^2}\right\} + \frac{1}{2}\left\{\frac{\Delta.r^2}{l_2^2}\right\} + \frac{1}{2}\left\{\frac{\Delta.r^2}{l_3^2}\right\} + \ldots\ldots\ldots + \frac{1}{2}\left\{\frac{\Delta.r^2}{l_n^2}\right\}$$

Then it is easily seen that

$$\frac{C}{8} = S_1 + S_2$$

And also, $S_1 - S_2 = \frac{1}{2}\left\{\frac{\Delta.r^2}{r^2}\right\} - \frac{1}{2}\left\{\frac{\Delta.r^2}{l_n^2}\right\}$

But $l_n^2 = 2.r^2$

So $S_1 - S_2 = \frac{1}{2}\left\{\Delta - \frac{\Delta}{2}\right\} = \frac{1}{2}\left\{\frac{\Delta}{2}\right\} = \frac{\Delta}{4} \approx 0 \quad \because \Delta \to 0$

$\therefore S_1 = S_2$

And hence $\frac{C}{8} = S_1 + S_2 = 2S_2$

$$\frac{C}{8} = \left\{\frac{\Delta.r^2}{l_1^2}\right\} + \left\{\frac{\Delta.r^2}{l_2^2}\right\} + \left\{\frac{\Delta.r^2}{l_3^2}\right\} + \ldots\ldots\ldots + \left\{\frac{\Delta.r^2}{l_n^2}\right\}$$

If we closely examine the above we can understand that since $l_1 \approx r$, the denominator of the first term becomes l_1^2. Similarly since, $l_2 \approx l_1$, denominator of second term becomes, l_2^2. But if this is put forth as a basis for deriving the above expression, it will lead to a mathematical fallacy. The fallacy is that if $l_1 \approx r$ and $l_2 \approx l_1$ etc r will apparently become equal to l_n which is $\sqrt{2}r$. To avoid this fallacy only, Jyeshtadeva has introduced a concept of limits, as shown below.

When $(l_2 - l_1) \to 0$, $l_1 l_2 \to \frac{1}{2}\{l_1^2 + l_2^2\}$ and $\frac{1}{l_1 l_2} = \frac{1}{2}\left\{\frac{1}{l_1^2} + \frac{1}{l_2^2}\right\}$

And logically it can be proved that

$$\frac{C}{8} = \frac{1}{2}\left\{\frac{\Delta r^2}{r^2} + \frac{\Delta r^2}{l_1^2}\right\} + \frac{1}{2}\left\{\frac{\Delta r^2}{l_1^2} + \frac{\Delta r^2}{l_2^2}\right\} + \frac{1}{2}\left\{\frac{\Delta r^2}{l_2^2} + \frac{\Delta r^2}{l_3^2}\right\} + \ldots + \frac{1}{2}\left\{\frac{\Delta r^2}{l_{(n-1)}^2} + \frac{\Delta r^2}{l_n^2}\right\}$$

$$= \left\{\frac{\Delta.r^2}{l_1^2}\right\} + \left\{\frac{\Delta.r^2}{l_2^2}\right\} + \left\{\frac{\Delta.r^2}{l_3^2}\right\} + \ldots\ldots\ldots + \left\{\frac{\Delta.r^2}{l_n^2}\right\}$$

Proceeding further,

Since

$l_1^2 = r^2 + \Delta^2$; $l_2^2 = r^2 + [2\Delta]^2$; $l_3^2 = r^2 + [3\Delta]^2$; $l_4^2 = r^2 + [4\Delta]^2$; ……
$l_{(n-1)}^2 = r^2 + [(n-1)\Delta]^2$; $l_n^2 = r^2 + [(n\Delta)]^2$, *where* $n.\Delta = r$ *and* $l_n = 2r^2$

and since $\frac{C}{8} = \left\{\frac{\Delta.r^2}{l_1^2}\right\} + \left\{\frac{\Delta.r^2}{l_2^2}\right\} + \left\{\frac{\Delta.r^2}{l_3^2}\right\} + \ldots\ldots\ldots + \left\{\frac{\Delta.r^2}{l_n^2}\right\}$

$$\frac{C}{8} = \left\{\frac{\Delta r^2}{(r^2 + \Delta^2)}\right\} + \left\{\frac{\Delta r^2}{(r^2 + (2\Delta)^2)}\right\} + \left\{\frac{\Delta r^2}{(r^2 + (3\Delta)^2)}\right\} + \ldots\ldots + \left\{\frac{\Delta r^2}{(r^2 + (n\Delta)^2)}\right\}$$

Each of the terms above on the right-hand side can be expanded into infinite series $\because \left|\frac{\Delta^2}{r^2}\right| \le 1$

$$\left\{\frac{\Delta r^2}{(r^2 + \Delta^2)}\right\} = \frac{\Delta r^2}{r^2\left(1 + \frac{\Delta^2}{r^2}\right)} = \frac{\Delta}{\left(1 + \frac{\Delta^2}{r^2}\right)} = \Delta\left\{1 - \left(\frac{\Delta^2}{r^2}\right) + \left(\frac{\Delta^2}{r^2}\right)^2 - \left(\frac{\Delta^2}{r^2}\right)^3 + \ldots\ldots\right\}$$

From the original text, the expression $\frac{p}{(a+x)}$ is split into

$\frac{p}{(a+x)} = \frac{p}{a} - \frac{p.x}{a(a+x)}$ and p is called dividend; $(a+x)$ is called divisor; and "a" is termed as "*gunakara*" and "x" is termed as '*gunaharantharam*". Noting that $\frac{p}{(a+x)} = \frac{p}{a} - \frac{px}{a(a+x)}$ we can derive an infinite series when x < a

by expanding $\frac{px}{a(a+x)}$ in the same manner, $\frac{p}{a}\left\{\frac{x}{(a+x)}\right\}$ becomes $\frac{p}{a}\left\{\frac{x}{a} - \frac{ax}{(a+x)}\right\}$

$$\frac{p}{(a+x)} = \frac{p}{a}\left\{1 - \frac{x}{a} + \left(\frac{x}{a}\right)^2 - \left(\frac{x}{a}\right)^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\right\}$$

Similarly, it can be shown that,

$$\frac{p}{(a-x)}=\frac{p}{a}\left\{1+\frac{x}{a}+\left(\frac{x}{a}\right)^2+\left(\frac{x}{a}\right)^3+\ldots\ldots\ldots\ldots\ldots\right\}$$ In the original text, we can notice certain observations about infinite series expansions while explaining the above. It is said that terms in these expansions will never end and they only become smaller and smaller as number of terms increase. Thus depending up on the desired accuracy we can limit the number of terms by excluding further terms. From this statement it could be understood that the mathematicians of the Kerala School had adequate knowledge regarding the 'convergent' property of series expansions. These expansions are discussed in detail in the next chapter of the *Yukthibhasa*, with numerical examples.

Thus

$$\frac{\Delta.r^2}{\left(r^2+\Delta^2\right)}=\Delta\left\{1-\left(\frac{\Delta^2}{r^2}\right)+\left(\frac{\Delta^2}{r^2}\right)^2-\left(\frac{\Delta^2}{r^2}\right)^3+\ldots\ldots\ldots\right\}$$

$$\frac{\Delta r^2}{\left(r^2+(2\Delta)^2\right)}=\Delta\left\{1-\left(\frac{(2\Delta)^2}{r^2}\right)+\left(\frac{(2\Delta)^2}{r^2}\right)^2-\left(\frac{(2\Delta)^2}{r^2}\right)^3+\ldots\ldots\right\}$$

$$=\Delta\left\{1-\left(\frac{(3\Delta)^2}{r^2}\right)+\left(\frac{(3\Delta)^2}{r^2}\right)^2-\left(\frac{(3\Delta)^2}{r^2}\right)^3+\ldots\ldots\ldots\right\}$$

$$\frac{\Delta r^2}{\left(r^2+(n\Delta)^2\right)}=\Delta\left\{1-\left(\frac{(n\Delta)^2}{r^2}\right)+\left(\frac{(n\Delta)^2}{r^2}\right)^2-\left(\frac{(n\Delta)^2}{r^2}\right)^3+\ldots\ldots\right\}$$

By summing up all the terms in LHS, we have C/8 and thus

$$\frac{C}{8}=\{\Delta(1+1+1+\ldots\ldots+1)\}-\left(\Delta\left(\frac{\Delta^2}{r^2}+\frac{(2\Delta)^2}{r^2}+\frac{(3\Delta)^2}{r^2}+\ldots\ldots+\frac{(n\Delta)^2}{r^2}\right)\right)+$$

$$\left(\Delta\left\{\left(\frac{\Delta^2}{r^2}\right)^2+\left(\frac{(2\Delta)^2}{r^2}\right)^2+\left(\frac{(3\Delta)^2}{r^2}\right)^2+\ldots\ldots\ldots\ldots+\left(\frac{(n\Delta)^2}{r^2}\right)^2\right\}\right)-$$

$$\left(\Delta\left\{\left(\frac{\Delta^2}{r^2}\right)^3+\left(\frac{(2\Delta)^2}{r^2}\right)^3+\left(\frac{(3\Delta)^2}{r^2}\right)^3+\ldots\ldots\ldots\ldots+\left(\frac{(n\Delta)^2}{r^2}\right)^3\right\}\right)+$$

$$\left(\Delta\left\{\left(\frac{\Delta^2}{r^2}\right)^n+\left(\frac{(2\Delta)^2}{r^2}\right)^n+\left(\frac{(3\Delta)^2}{r^2}\right)^n+\ldots\ldots\ldots\ldots+\left(\frac{(n\Delta)^2}{r^2}\right)^n\right\}\right)$$

Examining each term in RHS, we have the following.

<u>1st Term.</u>

$$\{\Delta(1+1+1+\ldots\ldots\ldots\ldots\ldots\ldots+1)\}=n.\Delta=r$$

Here $n.\Delta=r$ since we originally divided r into n parts

<u>2nd Term.</u>

$$\left(\Delta\left(\frac{\Delta^2}{r^2}+\frac{(2\Delta)^2}{r^2}+\frac{(3\Delta)^2}{r^2}+\ldots\ldots\ldots\ldots\ldots\ldots+\frac{(n\Delta)^2}{r^2}\right)\right)$$

$$=\frac{\Delta^3}{r^2}\{1^2+2^2+3^2+\ldots\ldots\ldots\ldots\ldots\ldots n^2\}$$

$$=\frac{\Delta^3}{r^2}\left\{\frac{n(n+1)(2n+1)}{6}\right\}$$

When $n\to\infty$, $(n+1)\approx n$ and $(2n+1)\approx 2n$ so that

$$\frac{\Delta^3}{r^2}\left\{\frac{n(n+1)(2n+1)}{6}\right\}=\frac{\Delta^3}{r^2}\left\{\frac{2n^3}{6}\right\}=\frac{\Delta^3}{r^2}\frac{n^3}{3}=\frac{(\Delta n)^3}{3,r^2}=\frac{r^3}{3.r^2}=\frac{r}{3}$$

<u>3rd Term</u>

By a similar approach,

$$\left(\Delta\left\{\left(\frac{\Delta^2}{r^2}\right)^2+\left(\frac{(2\Delta)^2}{r^2}\right)^2+\left(\frac{(3\Delta)^2}{r^2}\right)^2+\dots\dots+\left(\frac{(n\Delta)^2}{r^2}\right)^2\right\}\right)=\frac{r}{5}$$

In the next chapter of the *Yuktibhasa*, this summation is further elaborated, and generalized as *Sankalitham* (or Integrals).

When $n \longrightarrow \infty$,

$$\Delta(\Delta+2\Delta+3\Delta+\dots\dots n\Delta)=\frac{\Delta.n.(n+1).\Delta}{2}\rightarrow\frac{(n\Delta)^2}{2}=\frac{r^2}{2}$$

LHS is named as '*moola sankalitha*' of r which is equal to $\frac{r^2}{2}$

$$\Delta\left(\Delta^2+(2\Delta)^2+(3\Delta)^2+\dots\dots\dots\dots(n\Delta)^2\right)\rightarrow\frac{(n\Delta)^3}{3}=\frac{r^3}{3}\text{ LHS}$$

is named as "*varga sankalitha*" of "r" which is equal to $\frac{r^3}{3}$

In general

$$\Delta\left(\Delta^n+(2\Delta)^n+(3\Delta)^n+\dots\dots\dots(n\Delta)^n\right)\rightarrow\frac{(n\Delta)^{(n+1)}}{n+1}=\frac{r^{(n+1)}}{n+1}$$

LHS is the "*sankalitha*" of r^n which is equal to is $\frac{r^{(n+1)}}{(n+1)}$

Probably this was the first time in the history of mathematics that such a treatment of summation of terms extending infinitely was derived. The credit of this discovery is given to the Kerala mathematician and founder of the **Kerala**

School, Sangamangrama Madhava (c. 1340-1425). In the chapters that follow in the *Yuktibhasa*, the concepts of integration as well as differentiation are discussed further.

The great achievement of the Kerala School between the 14th to 16th centuries is that it proceeded from finite computational methods that were previously in use in India to an introduction of infinitesimal analysis, indeed a veritable 'passage to infinity'

Thus

$$\frac{C}{8}=r-\frac{1}{r^2}.\frac{r^3}{3}+\frac{1}{r^2}.\frac{r^5}{5}-\dots\dots\dots\dots\dots\dots=r-\frac{r}{3}+\frac{r}{5}-\dots$$

Since $r=\frac{D}{2}$

$$C=4.D-\frac{4D}{3}+\frac{4D}{5}-\dots\dots\dots\dots$$

$$\therefore\frac{C}{4D}=1-\frac{1}{3}+\frac{1}{5}-\dots\dots\dots$$

Thus we arrive at the relationship between Diameter (D) of the circle and the circumference (C), as indicated by the following verse in Tantrasangraha of Nilakantha

"Vyase Varidhi Nihate Roopa Hruthe
Vyasa Sagarbhihatae
Trisaradi Vishama Sanghya Bhakthamrunam
Swam Prathak Kramat Kuryat".

This may be translated as: "Multiply Diameter by four and then divide it by Unity, Multiply the diameter by four divide it by odd numbers, three, five etc then in order, separately deduct from each."*

Or in other words,

$$C = \pi.D = \frac{4D}{1} - \left(\frac{4D}{3} - \left(\frac{4D}{5} - \left(\frac{4D}{7} \ldots\ldots\ldots\ldots\right)\right)\right)$$

In the concluding part of the translated section it is suggested that instead of alternate plus terms and minus terms, all the positive terms and negative terms can be added separately and then from the sum of the positive terms the sum of negative terms could be deducted to get the desired result. Or in other words,

$$\frac{C}{8} = \left(r + \frac{r}{5} + \frac{r}{9} + \ldots\ldots\ldots\right) - \left(\frac{r}{3} + \frac{r}{7} + \frac{r}{11} + \ldots\ldots\right)$$

Thus, the *Yuktibhasa* not only provides the infinite expansion for π but also the derivation of the series, with the rationale given at each step. The method of derivation of infinite series expansion, when critically examined, may be understood as the 'geometrical differentiation' of the term $r\tan^{-1}x$ and then integrating to get the desired series expansion. In short, it is equivalent to $\int_0^1 \frac{d}{dx}(\tan^{-1}x).dx = \int_0^1 \frac{1}{(1+x^2)}.dx$ which eventually produces the so-called Gregory series expansion when $x \le 1$. And thus, proceeding in similar steps, in the subsequent sections of Chapter 6 of the *Yuktibhasa,* the derivations of the general expression for $\tan^{-1}x$, where $x \le 1$ and other different series expansions for π are derived. And given the slow convergence of the π series, methods to truncate infinite series expansions using 'remainder' terms are also discussed. It is clear that once the logical approach adopted in the *Yuktibhasa* is understood, the title of the book translated as the " language of rationale " would seem very apt.

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2. *Ganitha Yukti Bhasa*: Printed Version of MS in Madras Oriental Library
3. *Yukti Bhasa*: Palm Leaf Manuscript Tripuithura Manuscript Library
4. *Manushyalaya MahaCandrika* of Thirumangalath Nilakantha: (Malayalam Interpretation by K. Nilakanthan Asari)
5. Plane Trigonometry (Part 1 &2): S.L Loney

NOTES

The sources of the "*original text*" referred to in the literal translation above are two printed versions of the original palm leaf manuscripts of *Yuktibhasa* available in Malayalam.

1. One is the printed by Managalodam Press Trichur (1948) with footnotes and explanations by Late RamaVarma Maruthampuran and Late A.R.Akhilaswra Aiyer. This is available only for the mathematical part of the book. For transcribing this text it is said that four different palm leaf manuscripts were used.

   - Tripunithura Sanskrit College –MS Library
   - Kooli Variyam- Private family collection
   - Desamangalam Mana-
   - Kodungalloor Kovilakam

   [Out of the above collections, while preparing the above paper, I located the MS in Tripunithura Sanskrit College MS library and have verified the content. The exact locations where the other Manuscripts are now available are yet to be investigated.]

2. The other printed version of the *Yuktibhasa* is one, which was published from Madras Oriental Manuscript Library nearly fifty years ago. It is based on the MS available in Madras Manuscript

Library in Chennai. Content of printed version is one and the same as that of Mangalodayam except for many 'printer's devils'. This was just a printed version of the MS and not at all edited or corrected by an expert. Still it is in very much in agreement with the printed version mentioned under (1). For preparation of (1), this MS was not referred to. Thus the agreement in content of different MS from different sources indicates the authenticity as well as the purity of the original text.

6

MATHEMATICS OF THE *TANTRASAṄGRAHA* AND ITS SUCCESSORS

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INTRODUCTION

The *Tantrasaṅgraha*, (or 'compendium of science')[1] is one of the major works of Nilākaṇṭha Somayāji, a learned Nampūtiri brahmin who lived (14 June 1444 - ca.1545) in the village of Tṛkkaṇṭiyur, near Tirur, a reputed seat of learning in South-Malabar. His gurus were Ravi (*Vedānta*, general *Jyotiṣa*) and Dāmodara (Mathematics, Astronomy), son of the famous Parameśvara, founder of the *Dṛgganita* type of Kerala astronomy.

The *Tantrasaṅgraha*, composed in 1500, is a comprehensive and erudite treatise on astronomy, 432 verses divided into 8 chapters consisting of (1)The mean planets, (2)The true planets, (3) Gnomonic shadow (4).Lunar eclipses, ... etc.[2] This paper will be devoted to a part of the second chapter, in which Nilākaṇṭha dealt, in the manner already set by some of his predecessors, with the problem of constructing an accurate table of values related to what we would now call trigonometric functions, such as sine, cosine, versed sine, etc.

As a first successor of the *Tantrasaṅgraha*, we will consider the *Yuktibhāṣā*,[3] since we had the opportunity to work, as part of

the AHRB project, on an English translation of this work made by K.V.Sarma.[4]

The *Yuktibhāṣā* is not a commentary on the *Tantrasaṅgraha*, but rather "its aim is to provide the basic equipment needed by one who desires to study the computation of planetary movements as depicted in the *Tantrasaṅgraha*. This purpose it serves by introducing the basic concepts and theories of mathematics and astronomy, providing the definitions, and setting out the methodologies and their rationales."[5]

The *Yuktibhāṣā* is a very close successor of the *Tantrasaṅgraha*, since its author[6] Jyeṣṭhadeva (A.D. ca 1500-1610), a Nampūtiri Brahmin of the Parakroḍa family (Paraṅnoṭṭu Nampūtiri in Malayalam), was a younger contemporary of Nilākaṇṭha[7], whom he also mentions as 'Ācārya' (teacher).[8] In fact, Jyeṣṭhadeva was also, like Nilākaṇṭha, a disciple of Dāmodara.[9]

THE CIRCLE

In order to introduce the matter, let us now describe succinctly the content of *Yuktibhāṣā*'s seventh chapter entitled 'Derivations of Sines':

A circle of circumference 21600 (= 360×60 as the number of minutes in a sexagesimal system of measuring the angles) is constructed with its North-South ('horizontal') and East-West ('vertical') diameters. (See Figure I below) Two equilateral triangles, NAO and OBS, are constructed, and their symmetry verified with the help of the plumb line, i.e. the two straight lines NA and SB should be prolonged beyond E so that a plumb line (also called *lamba* 'perpendicular') hanged from their intersection would fall exactly on O. Since the two perpendiculars from A and B to the NS diameter fall on two points A' and B' cutting each half-diameter into two equal segments, the distance between A and B is equal to the length of the half-diameter, which also equals the side length of the two equilateral triangles. By constructing two equilateral triangles in the western part of the circle, one obtains thus a regular inscribed hexagon NABSCD.

Now, one can define the 'full-sine' (*samasta-jyā*) of two *rāśi*[10] (AB), which is equal to the radius of the circle. A full-sine, which is a complete chord, must not be confused with the 'half-

sine' (*ardha-jyā*), often connoted 'sine' (*jyā*), which is the half-chord (AF for an arc of one *rāśi*).

Figure I

It is plain to see that these 'sines' differ from our sines, which are defined in a circle of radius unity. In order to emphasize this difference, we will indicate '*r*sines' as the Indian sines where *r*sine $a = r \times \text{sine}\,(a/r)$.

The '*r*versed-sine' (*śara*, an arrow) is then defined as the extension (EF for an arc of one *rāśi*) from the middle of the full-*r*sine to the middle of the full arc, and the '*r*cosine' as the extension from the middle of the full-*r*sine to the center of the circle (OF for an arc of one *rāśi*).

There is a relation between the *ardha-jyā*, the *śara* and the chord of one *rāśi*, that is

$([r\text{sine}\ (1\ r\bar{a}\acute{s}i)]^2 + [\acute{s}ara\ (1\ r\bar{a}\acute{s}i)]^2)^{1/2} = \sqrt{AF^2 + FE^2}$ = full *r*sine (1 *rāśi*) = AE

Placing now AE symmetrically to the East-West line, one observes that its half, A"F', which is the "half-*r*sine" of 1/2 *rāśi*, allows to find the *r*cosine of 1/2 *rāśi* by a relation of the same type $\sqrt{A''O^2-(AE/2)^2}$ = OF' = *r*cos (1/2 *rāśi*) = rsine (2.5 *rāśi*), and finally *śara* (1/2 *rāśi*) = OE - OF' = *r* - *r*cos (1/2 *rāśi*).

The author notes that this method enables to derive the *r*sines of successive (halved) arcs with the help of squares and square roots only.

CONSTRUCTION OF A TABLE

The circle is now divided into four quadrants (*pāda*), each of them divided into 24 equal arc-bits of 1/8 *rāśi* each (corresponding to the angle 225'), by 25 points (A_0 = E, A_1, A_2, etc.) on the circle.

From each point, the horizontal segment leading to the East defines the *r*sine of the corresponding arc, and the vertical segment leading to the North line defines the *r*cosine (for instance, B_1A_1 = *r*sine (1/8 *rāśi*) = sine n°1). The East and North lines are the bases of the *r*sines and *r*cosines respectively, and their tips are the dividing points.

Each arc-bit has also its base and tip, but, for practical reason, they are defined differently according to the *r*sines and *r*cosines. The *r*sine base of an *arc-bit* is its end nearest to the East line, and its *r*sine tip is its end furthest to the East line. It is the contrary for the *r*cosine.

All the segments defined above, in order to visualize the different *r*sines and *r*cosines, define on each other sub-segments, of which the closest to the circle are called *bhuja-khaṇḍa* ('*r*sine differentials') and *koṭi-khaṇḍa* ('*r*cosine-differentials'). For instance, the first *bhuja-khaṇḍa* is the *r*sine of the first arc-bit, that is *r*sine n°1 itself (B_1A_1), the second *bhuja-khaṇḍa* is that part of the second rsine from the tip of the second *r*sine to the *r*cosine of the first arc-bit (that is H_3A_2), etc.

It must be noted that these differentials form the lateral sides of right-angled triangles, of which hypotenuses are all equal to the full

chord of the common arc-bit. We thus have 24 right-angled triangles, with equal hypotenuses but different lateral sides.

A certain arc, measured from E to a point in the first quadrant, is called *iṣṭa-bhuja-cāpa* ('arc of which *r*sine is such'). A certain arc, measured from N, is called *iṣṭa-koṭi-cāpa* ('arc of which *r*cosine is such')[11].

One can compute the sides of the little triangles having same hypotenuse (225'), which are the *r*sine and *r*cosine differentials, and tabulate them (*paṭhita-jyā*), as in earlier texts.[12]

If a desired point is at the end of an arc (multiple of 225'), its *r*sine will be in the table. If not, one has to add the proportionate part of the increase to the next end of arc. But, a simple method, as the rule of three, doesn't work, for if 2^{nd} arc = 2×1^{st} arc, 3^{rd} arc = 3×1^{st} arc, it is not true for the *r*sines. In fact, it will be true for the first arcs, almost flat because the *śara* are minutes, but not for the following ones, more curved when the *śara* increase.

Recursion Method To Compute The Tabular Sines

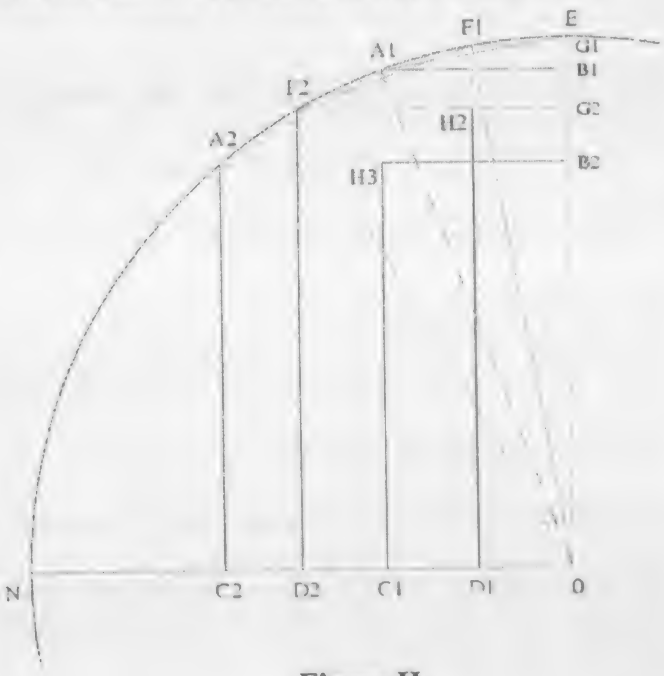

Figure II

Refer to Figure II. In order to compute accurate tabular *r*sines, one draws the 225' arc-bit symmetrically with respect to the EW line (F_1E = 112.5) so that F_1G_1 = *r*sine n°0.5 = *r*sine 112.5' ≈ 112.5, and the same arc from E (A_1E = 225). One has *r*cos 112.5' = $\sqrt{r^2 - 112.5^2}$ = *r*sine n°23.5 and *r* - *r*sine arc n°23.5 = *śara* of the previous *r*sine (112.5')

Comparing different similar triangles, the author now obtains the other tabulated *r*sines and *r*cosines:

The similarity between the triangles A_1B_1E and OG_1F_1 yields:

rsine n°1 = A1B1 = (OG1/OF1)×A1E = ([rcos n°0.5]/r)×225
= ([rsine n°23.5]/r)×225

and

rsine n°1 = rsine n°23 = A1C1 = EO - EB1
= EO - (F1G1/OF1)×A1E
= *r* - ([*r*sine n°0.5]/*r*)×A_1E = *r* - ([*r*sine n°0.5]/*r*)×225

The similarity between the triangles $F_2H_2F_1$ and OB_1A_1 yields:

*r*sine n°1.5 = F_2G_2 = H_2G_2 + F_2H_2 = H_2G_2 + (A_1C_1/A_1O)×F_2F_1
= *r*sine n°0.5 + ([*r*cos n°1]/*r*)×F_2F_1 = *r*sine n°0.5 + ([*r*cos n°1]/*r*)×F_2F_1
= 112.5' + ([*r*sine n°23]/*r*)×225, and *r*cos n°1.5 = *r*sine n°22.5
= F_2D_2 = F_1D_1 - F_1H_2
= F_1D_1 - (A_1B_1/A_1O)×F_2F_1
= *r*cos n°0.5 - ([*r*sine n°1]/*r*)×F_2F_1 = *r*cos n°0.5 - ([*r*sine n°1]/*r*)×225
= *r*sine n°23.5 - ([*r*sine n°1]/*r*)×225

The similarity between the) triangles $A_2H_3A_1$ and OG_2F_2 yields :

*r*sine n°2 = A_2B_2 = B_2H_3 + H_3A_2 = B_2H_3 + (G_2O/OF_2)×A_2A_1
= *r*sine n°1 + ([*r*cos n°1.5]/*r*)×A_2A_1 = *r*sine n°1 + ([*r*cos n°1.5]/*r*)×225
= *r*sine n°1 + ([*r*sine n°22.5]/*r*)×225,

And

*r*cos n°2 = *r*sine n°22 = A_2C_2 = A_1C_1 - A_1H_3 = A_1C_1 - (F_2G_2/OF_2)×A_2A_1
= *r*cos n°1 - ([*r*sine n°1.5]/*r*)×A_2A_1 = *r*cos n°1 - ([*r*sine n°1.5]/*r*)×225
= *r*sine n°23 - ([*r*sine n°1.5]/*r*)×225

One can thus, beginning from *r*sine n°0.5 and *r*cos n° 0.5 = *r*sine n°23.5, tabulate all the rsines n°i and n°[i + 0.5] (i = 1 to 23), and then reject the n°[i+0.5] if necessary.

Translated into modern symbolism, we have the following recurrent formulas :

*r*sine (*N*×112.5') = *r*sine ([*N* -2]×112.5') + [*r*cos ([*N*-1]×112.5')×225]/*r*

and

*r*cos (*N*×112.5) = *r*cos ([*N* -2]×112.5') - [*r*sine ([*N*-1]×112.5')×225]/*r*,

where *N* is an integer between 2 and 48, and *r* ≈ 3437.75.

For *N* = 1, we take *r*sine 112.5' = 112.5 and *r*cos 112.5' = $\sqrt{r^2 - 112.5^2}$ ≈ 3435.91

The first values obtained by this process (and their modern equivalents) are:

N	*angle corresponding to arc (N×112.5')*	*rsine (N×112.5')*	*sine (modern)*	*rcos (N×112.5')*
1	1°45'	112.5	0.032725 (0.032719)	3435.91
2	3°45'	224.88	0.065415 (0.065403)	3430.39
3	5°37.5'	337.02	0.098034 (0.098017)	3421.03
4	7°30'	448.79	0.130549 (0.130526)	3408.33
5	9°22.5'	560.09	0.162924 (0.162895)	3391.66
6	11°15'	670.77	0.195125 (0.195090)	3371.67
7	13°0.5'	780.77	0.227116 (0.227076)	3347.91
8	15°	889.91	0.258864 (0.258819)	3320.57
9	16°52.5'	998.10	0.290335 (0.290285)	3289.67
10	18°45'	1105.22	0.321495 (0.321439)	3255.24

Other Recursion Formulas Involving the Rsine-Differentials

In the following section, the author is looking for a relation between the rsine of a certain arc-bit and the second differential of rsines. Put into modern notations, his text is equivalent to:

$$[r\cos([N\text{-}1]\times 112.5') - r\cos([N+1]\times 112.5')].\frac{r}{225}$$

$= [\text{rsin}(N\times 112.5') - \text{rsin}([N\text{-}2]\times 112.5')] - [\text{rsin}([N+2]\times 112.5') - \text{rsin}(N\times 112.5')]$,
which results from formulae explained above.

And, since $\text{rcos}([N\text{-}1]\times 112.5') - \text{rcos}([N+1]\times 112.5']$

$$= \text{rsin}\ (N\times 112.5').\ \frac{r}{225}$$

by the same formulae, one has equality to:

$$\text{rsin}\ (N\times 112.5').\left(\frac{r}{225}\right)^2\ [\text{rsin}(N\times 112.5') - \text{rsin}([N\text{-}2]\times 112.5')]$$
$$- [\text{rsin}([N+2]\times 112.5' - \text{rsin}(N\times 112.5')],$$

which is expressed in the text as:

> "When the first sine is multiplied by the square of the full sine of the arc-bit and divided by the square of the *trijyā*,[13] the result would be the difference between the first sine differential and the second differential".[14]

Finding The *R* sine of An Arc which is not Tabulated

As we have seen, our author has already alluded to this problem, but, in a section preceding the last one, concerned with the *r*sine-differences, he gives the following method (put into modern notation. (See Figure III)

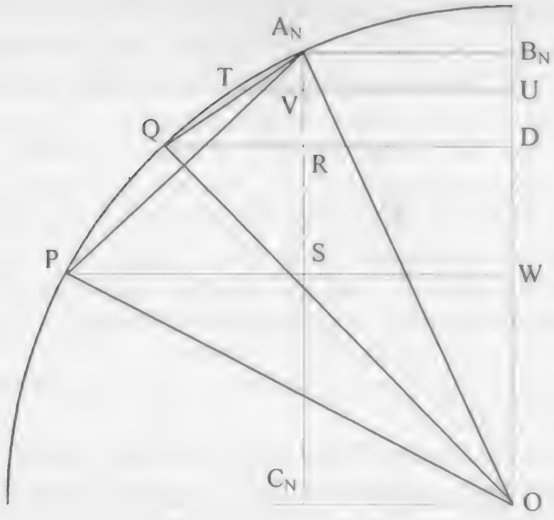

Figure III

First, one takes the tabular point A_N nearest to P and define the arc PA_N as *śiṣṭa-cāpa* ('remaining arc' a). By a rule of three based on similar triangles PA_NS and OQD (see the Figure), one has $PA_N\times QD/OQ = A_NS$ and $PA_N\times DO/OQ = PS$ (1).

But QD and DO are unknown, so that the following method is given:

One takes the chord QA_N as hypotenuse and its *r*sine and *r*cosine as *icchāphala's* to be "applied" to the relevant tabular rsines. This could be understood as calculating

$QD = DR + RQ = A_NB_N + RQ$,

$DO = OB_N - DB_N = A_NC_N - RA_N$

and, since RQ and RA_N are unknown (but small), take $QD \approx A_NB_N$ and $DO \approx A_NC_N$. Finally, this method yields $PA_N\times A_NB_N/OQ = A_NS$ and $PA_N\times A_NC_N/OQ = PS$.

So, in modern symbolism foı sine :

*r*sin $(N\times 112.5' + a) = PW = SW + PS = A_NB_N + PA_N\times DO/OQ \approx A_NB_N + PA_N\times A_NC_N/r$,

Or $r\sin(N\times 112.5' + a) = r\sin(N\times 112.5) + a\times r\cos(N\times 112.5')/r$

The author concludes by saying that, if some more accuracy is needed, one can take one fourth of the *śiṣṭa-cāpa*, or even the half of it, as the full chord, quoting a verse beginning by *iṣṭadoḥkoṭidhanuśoḥ* attributed by K.V. Sarma to the astronomer Mādhava. This probably means that one has to compute more accurate values of QD and OD, than A_NB_N and A_NC_N respectively, by equating TU and UO (see the Figure) to A_NB_N and A_NC_N respectively, and to replace them ultimately in the equations (1).

This method yields:

$r\sin(N\times 112.5' + a) = A_NB_N + PA_N\times DO/OQ$, where DO, instead of being replaced by A_NC_N, is equated to $A_NC_N - RA_N$, with $RA_N = QA_N\times TU/OT$ (since RQA_N and UOT are two similar triangles)

Thus

$$r\sin(N\times 112.5' + a) = A_NB_N + PA_N\times(A_NC_N - RA_N)/OQ$$
$$= A_NB_N + PA_N\times(A_NC_N - QA_N\times TU/OT)/OQ$$
$$\approx A_NB_N + PA_N\times(A_NC_N - QA_N\times A_NB_N/OT)/OQ,$$

Or

$$r\sin(N\times 112.5' + a) = r\sin(N\times 112.5') + a\times[r\cos(N\times 112.5') - (a/2)\times r\sin(N\times 112.5')/r]/r$$

By saying that this process could be continued, the author probably indicates an expansion of $r\sin(N\times 112.5' + a)$, from the values of $r\sin(N\times 112.5')$ and $r\cos(N\times 112.5')$, which is equivalent to the Taylor expansion

$\sin(N\times 112.5' + \alpha) = \sin(N\times 112.5') + \alpha\times\cos(N\times 112.5') - \frac{1}{2}\alpha^2\times\sin(N\times 112.5')$, where $\alpha = a/r$.

We find the verse alluded to by the *Yuktibhāṣā* in the *Tantrasangraha*, Chapter 2[15] :

[16]*iṣṭadoḥkoṭidhanuṣoḥ svasamīpa samīrite //10//*
jye dve sāvayave nyasya[17] *kuryādūnādhikaṁ dhanuḥ /*
dvighnatalliptikāptā eka-śara-śaila-śikhi-indavaḥ //11//

nyasyācchedāya ca mithastatsaṁskāravidhitsayā /
chitvaikāṁ prāk kṣipejjahyāt taddhanuṣyadhikonake //12//
anyasyāmatha tāṁ dvighnāṁ tathā 'syāmiti[18] *saṁskṛtiḥ /*
iti te kṛtasaṁskāre svaguṇau dhanuṣostayoḥ //13//

Translation:

Having placed two chords, made of parts[19], close by the searched *r*sine and *r*cosine [arcs][20], calculate the additional or missing arc.

Take as denominator one-five-seven-three-one (13751) divided by the minutes[21] of this (additional or missing arc) multiplied by two, in order to refine these (searched rsine and cosine) one after the other.

Having divided one (*r*cosine[22]), add or subtract (the result) from the preceding (*r*sine), depending on the additional or missing (character) of the arc.

The preparation is the same for this (last result) doubled (and added or subtracted) to the other (chord).

Two appropriate refinements are brought this way to the two [arcs][23].

To translate the first half of verse 13, we had recourse to the following commentary:

athaiva ṁ kṛtā ṁ tā ṁ dviguṇitā ṁ kṛtvā pūrvoktenaiva hārakeṇa vibhajya labdhaṁ yat phalaṁ tat punaranyasyā ṁ sādhyajyāyāmeva ta ṁ dhanuṣa ūnādhikavaśād ṛṇaṁ dhanaṁ vā kuryāt /	Now, having doubled this (quantity just) obtained, having divided (it) by the divisor, the result, which is a quotient, let him add it again to the other chord as a negative or positive quantity depending on the additional or missing character of the arc.

This is an extract from the *Laghuvivṛti*, a 'concise commentary' to the *Tantrasa ṅgraha*, composed by Śa ṅkara Vāriyar (ca.1500-1560), who also made a *Yuktidīpikā*, another commentary to the *Tantrasa ṅgraha*, and a *Kriyākramakarī*, an unfinished[24] commentary on Bhāskara's *Līlāvatī*. Śa ṅkara was

the pupil of Nīlakaṇṭha[25]. Śa ṅkara could also have received instruction from Citrabhānu (1475-1550)[26].

The operations are the following ones :

Supposing the rsines and rcosines of $N\times 112.5$ tabulated, we want to calculate

$r$sine ([$N\times 112.5$']+ a), where $0 < a < 112.5'$.

First, we compute $\dfrac{13751}{2\times a} = \dfrac{4\times 3437.5}{2\times a} = \dfrac{2\times 3437.75}{a} = \dfrac{2\times r}{a}$,

We divide $r$sin ($N\times 112.5$) by this number and subtract the result from $r$cos ($N\times 112.5$), obtaining:

$$r\cos(N\times 112.5') - [r\sin(N\times 112.5')]/\frac{2\times r}{a}$$

$$= r\cos(N\times 112.5') - \frac{a}{2\times r}\times r\sin(N\times 112.5')$$

$$= r\times[\cos(N\times 112.5') - \frac{1}{2}\times[\sin(N\times 112.5')]\times\alpha], \text{ where } \alpha = a/r$$

Then, we divide again this result, doubled, by $\dfrac{2\times r}{a}$, and add it to $r$sin ($N\times 112.5$') :

$$\text{rsin } (N\times 112.5') + 2[\text{rcos } (N\times 112.5') - [\text{rsin } (N\times 112.5')]/]/$$

$$= r\times\sin(N\times 112.5') + 2\times r\times[\cos(N\times 112.5') - \tfrac{1}{2}\times[\sin(N\times 112.5')]\times\alpha]\times\alpha/2$$

$$= r\times[\sin(N\times 112.5') + \cos(N\times 112.5')\times\alpha - \tfrac{1}{2}\times[\sin(N\times 112.5')]\times\alpha^2,$$

which contains the three first terms of the Taylor expansion of sin ($N\times 112.5' + \alpha$), as seen above.

The *Tantrasaṅgraha* yields thus :

$$\text{rsin } (N\times 112.5' + a) = r\times[\sin(N\times 112.5') + \cos(N\times 112.5')\times\alpha - \tfrac{1}{2}\times[\sin(N\times 112.5')]\times\alpha^2.$$

But, as we have seen, the *Yuktibhāṣā* proposes to continue the process further by dividing the arc a by 2, 4 ,etc.

We have an echo of this continuous process, quickly abandoned, in the *Laghuvivṛti* :

yadyapyatra tataḥ pūrva ṁ koṭijyārdhataḥ tenaiva hāreṇa labdhaṁ dorjyāyāḥ tataḥ pūrvaṁ dorjyācatura ṁśāt koṭijyāyā ṁ tataḥ pūrva ṁ koṭijyāṣṭamā ṁśato dorjyāyā ṁ tataḥ ṣoḍaśā ṁśataḥ koṭijyāyā ṁ ca labdhaphala ṁ kartavyameva /

tathāpi tasyālpatvādevopekṣitamiti mantavyam /

Even if, in that case, the sine is then obtained from the half-cosine through the divisor, then the result obtained from the fourth part of the sine must be put into the cosine, then (the result obtained) from the eighth part of the cosine (must be put) into the sine and then (the result obtained) from the sixteenth part (of the sine must be put) into the cosine, however it must be understood that (this is) ignored because of its minute character.

ENDNOTES

1 There are at least two editions of the *Tantrasaṅgraha* : *Tantrasaṅgraha of Nilākaṇṭha Somayāji with Yuktidīpikā and Laghuvivṛti of Śaṅkara*, critically edited by K.V.Sarma, V.V.B.I.S. & I.S., Hoshiarpur, 1977, and *The Tantrasaṅgraha, A Work on Gaṇita, by GārgyaKerala Nilākaṇṭha Somasutvan with Laghuvivṛti of Śaṅkara Vāriar*, published by S.K.Pillai, Trivandrum Sanskrit Series n°188, Trivandrum, 1958.

2 More details in K.V.Sarma, *Tantrasaṅgraha* ..., pp.xxiv-xxxvi.

3 Also called *Gaṇitanyāyasa ṅgraha*, according to K.V.Sarma, *A History of the Kerala School of Hindu Astronomy*, Vishveshvaranand Institute, Hoshiarpur, 1971, p.60.

4 The original *Yuktibhāṣā* is in Malayalam, but there exists a *Gaṇitayuktibhāṣā*, composed in Sanskrit. According to K.V.Sarma (*Tantrasaṅgraha* ..., p.xlvi) 'a close examination of the work has

now revealed that this is a not-so-reliable attempt at literally translating the *Yuktibhāṣā* by some later scholar who was neither well versed in Sanskrit nor properly equipped in the subject'.

5 K.V.Sarma, *Tantrasaṅgraha* ..., p.xlv.

6 The extant manuscripts of this work do not mention the name of its author, but K.V.Sarma succeeded in identifying him from external sources (see K.V.Sarma, ' *Jyeṣṭhadeva and his identification as the author of Yuktibhāṣā*', *Adyar Library Bulletin*, 22 (1958), pp.35-40).

7 K.V.Sarma, *A History of the Kerala School* ..., p.59.

8 K.V.Sarma, *Tantrasaṅgraha* ..., p.xliv.

9 This, as well as his family name and his autorship of the *Yuktibhāṣā*, is ensured by a Malayalam commentary on the *Sūryasiddhānta* preserved in Baroda (K.V.Sarma, *Tantrasa ṅgraha* ..., p.lxvi).

10 In astronomy, a *rāśi* is a sign of the zodiac or the extension of it, i.e. 1800, corresponding to our 30°.

11 These expressions are thus comparable to our arc sine (or $\sin^{-1}$) and arc cosine (or $\cos^{-1}$).

12 These texts are not explicitely mentioned.

13 Or *tri-rāśi-jyā*, 'sine of three zodiac signs', that is rsine 5400 (arc equivalent to 90°) = r.

14 I must mention that this relation between rsine and second differential of rsines, and other similar relations, have already been described by R.C.Gupta in his article "Early Indians on second order sine differences", *Indian Journal of History of Science*, 7 (1972), pp.81-86.

15 K.V.Sarma, *Tantrasa ṅgraha* ..., p.112, and S.K.Pillai, *The Tantrasaṅgraha*, ..., p.19.

16 Contrary to the text given by R.C.Gupta, 'Second order interpolation in Indian mathematics up to the fifteenth Century', *Indian Journal of History of Science*, 4 (1969), p.93, the text of the *Tantrasaṅgraha* edited in Trivandrum doesn't begin by the words tatrāha mādhavaḥ 'Thus spoke Mādhava', nor does the K.V.Sarma's edition. But it is possible that the other text, to which R.C.Gupta refers, i.e. Nilākaṇṭha's commentary on Āryabhaṭa, does. Anyway, the *Yuktibhāṣā* refers, as already noted, to the beginning of verse 10.

17 *'nyasya* for *anyasya* in the Trivandrum edition.

18 syām in the Trivandrum edition.

19 This precision, sāvayave, ensures that we are dealing with arc-bits, multiples of the same arc.

20 The dual *dhanuṣoḥ* here shows that the text is dealing with the two sine and cosine 'cords', so that 'arcs' would be an inappropriate translation. One can also translate 'close by the arc, of which sine and cosine are searched', replacing the dual by a singular.

21 *liptikā* = *liptā* = 'minute of arc', a word of Greek origin.

22 One has to choose one of the *jyā*, the other being processed in the following verse.

23 Same remark as for the *dhanuṣoḥ* in verse 10.

24 By Śaṅkara, but it has been completed by Nārāyaṇa, when he was only eighteen years old. He begins his supplementation by the words 'This commentary was composed last by Tṛkkuṭaveli Śaṅkara Vāriyar. I have heard it said by Paraṅṅoṭṭu that it was composed with great care at the instance of Āzhāñceri.' This ensures that the first author was indeed Śaṅkara and that Nārāyaṇa (ca.1540-1610) was a younger contemporary of Jyeṣṭhadeva, who is here referred to by his family name Paraṅṅoṭṭu.. The 'Āzhāñceri' mentioned is Nārāyaṇa Āzhvāñceri Tāmprakkal, religious head of the nampūtiri brāhmans, whom Nilākantha has referred to as his patron (*Līlāvatī of Bhāskarācārya with Kriyākramakarī of Śaṅkara and Nārāyaṇa*, critically edited by K.V.Sarma, Vishveshvaranand Institute, Hoshiarpur, 1975, p.xvii).

25 K.V.Sarma, *Tantrasaṅgraha* ..., p.lxiii.

26 K.V.Sarma, *Līlāvatī* ..., p.xxi.

• *Note* : In the verse, the literal meaning of "*varidhi*" and '*sagara*", is ocean but it denotes the number four derived from '*Chaturarnava*" There were two systems of representing numbers, which were very common in traditional texts. One represented numbers with alphabets (i.e., the Aryabhathan system or the later refinement the *katapayadi* system) and the other by relating to real objects or objects described in Epics (*Puranas*). The latter was known as "*Bhuta Sankhya*" system. The verse quoted above and in the last paragraph of the original text beginning with "*Vayase Varidhi Nihate*" is a summary of the derivation explained in detail.

PART – III

Establishing Transmissions and other Philosophical and Methodological Issues

7

ESTABLISHING TRANSMISSIONS: SOME METHODOLOGICAL ISSUES*

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1. The Problem of Transmission

It is generally assumed that modern calculus developed from ideas and techniques inspired by ancient Greek mathematics culminating with the method of exhaustion deployed by Archimedes (287-212 BC).[1] After a period of more than eighteen centuries these techniques were rediscovered in Europe with the translation of the works of Archimedes in the 16th century, and developed further by a chain of European mathematicians including Roberval (1602-1675), Cavalieri (1598-1647) and Fermat (1601-1665), culminating in the consolidation of calculus by Leibniz (1646-1716) and Newton (1643-1727). It takes for granted that no new developments took place between the time of Archimedes and the 17th century that could have a bearing on this history. However, it has now become evident that there were important developments in mathematics in India between the 14th and 16th centuries that anticipated many of the discoveries

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attributed to European mathematicians. These were made by what has now come to be described as the Indian Kerala School of mathematics. They were the achievements of a chain of thinkers including Madhava (1350-1425), Paramesvara (c. 1370-1460), Nilakantha Somayaji (1444-1544) and Jyesthadeva (c.1500-1575).

The Indian discoveries include numerical integration methods concerned with the sum of various types of mathematical series, and infinite series expansions of pi, sinx, cosx and inverse tanx. These achievements go beyond ancient Greek discoveries where we do not find any systematic study of infinite series, or infinite series representations for pi and other trigonometric functions. These additional achievements, so crucial to the development of modern calculus, were made in India before they were made in modern Europe.[2] They raise the question of whether Indian mathematics contributed to the emergence of modern calculus in Europe.

This issue has been highly contested since the discovery of the works of the Kerala School by Whish in 1835. However, not much progress was made for over a century until Indian historians and mathematicians began to examine the question beginning with Marar and Rajagopal in 1945 with a paper on the anticipation of the arctan series, now attributed to Gregory, by Madhava. It was followed by the studies of Sarma on Kerala astronomy and Gupta on trigonometry in the early 1970s. The development of the debate over the next few decades made it evident that the issue could be settled only by deciding whether the priority of the Indian discoveries, and the fact that routes of their transmission to Europe had come into existence after the Portuguese arrival in India in 1498, constituted sufficient conditions to conclude transmission had occurred.

Some historians, such as Needham, argue that it is reasonable to base claims for transmission on the observed fact that an idea or practice occurred in one place before it appeared in another, and routes of communication existed between them, because "the details of any transmission are difficult to observe" (Needham 1969a, p.83). Needham considered it incumbent on those who claimed independent discovery to demonstrate their case.[3] But there were others who argued that evidence for

possibility of transmission was not sufficient evidence for transmission, and demanded those making such claims to provide positive evidence for it. Progress soon ground to a halt in a stand-off between the two positions. Nevertheless these debates constituted an advance since they made it no longer possible to take for granted the Eurocentric assumption that there was no Indian influence on modern calculus.

In order to resolve this deadlock Joseph and Almeida undertook the project of investigating whether a more positive case for transmission can be made by going beyond appeal to criteria of priority of discovery and possibility of communication. They set out to investigate the matter by adopting what they call 'the legal approach' to establishing transmissions. This involved testing whether transmission could be demonstrated on the grounds of (1) motivation, (2) opportunity, (3) circumstantial evidence, and (4) documentary evidence. Their initial conclusion, although this research is ongoing, appears to be that the legal criterion makes possible a stronger case for transmission than one which can be made by mere appeal to priority and possibility of communication. However, they ultimately found that a positive case could only be made on grounds of motivation, opportunity and circumstantial evidence, because they could not unearth any documentary evidence for transmission in spite of arduous research in archival materials in India and Europe. However, given that the evidence so far makes it more plausible to assume transmission than no-transmission, they are optimistic that further research would uncover documentary evidence too.

In this paper I will treat the absence of documentary evidence as evidence that the process of transmission may have occurred in a way that did not leave the sort of documentary evidence being sought by Joseph and Almeida. In particular I would like to explore whether the Kerala discoveries may have reached Europe as a set of computing skills transmitted by Indian craftsmen to their Portuguese counterparts, rather than theoretical ideas exchanged between Indian and Portuguese intellectuals. This approach does not preclude the possibility that intellectual exchanges may also have occurred. However, if transmission occurred not in the form of theoretical exchanges,

but of computing techniques and skills, it becomes necessary to examine how we can strengthen the methodological application of the legal criterion to include such cases. In order to pursue this investigation I will begin by examining the case made so far by Joseph and Almeida.

2. The Legal Case for Transmission

Let us now examine how Joseph and Almeida establish their case for transmission by applying the legal criterion. Since my description will be based on a paper they and others wrote for an earlier conference under the rubric Aryabhata Group, I will refer to their collective identity as AG. According to AG there was strong motivation for the import of Indian mathematical knowledge because of the interest in better computing techniques which provided greater accuracy in navigation, calendar making and practical mathematics. Indeed, the importance of such techniques can be appreciated by the fact that many European governments instituted huge prizes for the discovery of accurate techniques of navigation – including the Spanish government in 1567 and 1598 and the Dutch in 1632 – and leading scientists, including Galileo, competed for them.[4] Given that Indian mathematics had already discovered some of the advanced mathematical techniques at the time Europeans were searching for them they would have been motivated to learn these should the opportunity arise.

The opportunity also presented itself to the Europeans as a result of the route they discovered to reach India in 1498 by rounding the Cape of Good Hope. Moreover, the Portuguese in India landed at Cochin in Kerala which was the heartland for the development of the new mathematical ideas. Even if initially the Portuguese may have not been able to understand the Indian mathematical literature they would have been in a much better position to do so after the arrival of Jesuit missionaries trained in mathematical, navigational and astronomical sciences. One example is Matteo Ricci, who who stayed in India for over three years before leaving for China. In one of his letters from India he writes that he was trying to find out about the calendar from Brahmins and Moors – i.e. Hindu and Muslim sources.[5]

AG also find circumstantial evidence that the opportunity made available was successfully exploited, because Indian mathematical ideas were transmitted to Europe. In particular they make their case on four pieces of circumstantial evidence. The first involves a challenge issued by Pierre Fermat to European mathematicians where he called for integer solutions to the indeterminate equation $x^2 - Ay^2 = 1$, where A is any nonsquare integer. They find it interesting that Fermat mentions in a letter Frenicle a special case of the problem: What is the smallest square multiplied by 61 with unity added that makes a square? The choice of the particular coefficient $A = 61$ by Fermat is significant because the indeterminate equation he set as a problem was 'identical' to that set by the Indian mathematician Brahmagupta (598-670) and solved by Bhaskara II (1114-1185) when he developed the *chakravala* method to provide a general solution to the equation. In his *Bijaganita*, where he describes his general approach, Bhaskara II gives as a solved example of the equation precisely the case of A = 61. This is not a trivial coincidence if we consider that the solution $x = 1{,}766{,}319{,}049$, $y = 226{,}153{,}980$ involve quite large numbers. This leads AG to conclude that the coincidences in the problem set by Fermat, and the special stress made by him to look for solutions where the coefficient A = 61, strongly indicate that "Fermat probably had access to some Indian mathematical texts like the *Bijaganita*".[6]

Their second piece of circumstantial evidence is the following. Fermat developed many of the main ideas of the calculus thirteen years before the birth of Newton in 1642. The approach he makes is quite similar to that adopted by the Kerala School and appeals to a general formula described by Jyesthadeva in his *Yuktibhasa* (circa 1550). Jyesthadeva describes a method of deriving the sine table which can also be found in the *Tantrasangraha* of his predecessor Nilakantha. Fermat uses the same method to evaluate the area under a parabola $y = x^k$, but it is also a general formula to solve diverse problems in calculus. The same formula was worked on, in one version or another, by Pascal, Roberval, Cavalieri and John Wallis – some of the early pioneers of discoveries that led to the calculus. Indeed, it is significant that shortly after contact with

India many European mathematicians start solving mathematical problems using a method known to Indian mathematicians.[7]

Another item of circumstantial evidence for suspecting the influence of Indian mathematics is the adoption of day numbers in the scientific specification of dates. This had been in use in Indian astronomy from the time of Aryabhata since it eliminated all ambiguity due to calendrical differences. It soon came to be used in all Indian astronomical texts. However, they were only introduced into Europe by Julius Scaliger in 1582, in what is now known as the Julian day number system. It differs from the Indian day number system – the *Ahargana* – by beginning its count from the Biblical date of creation on 1 Jan – 4713 CE, instead of the start of the Kaliyuga on 17 Feb 3102 CE. AG suggests that it is no coincidence that the new system came to be adopted in Europe after contact with India when its advantage would have been immediately evident to Jesuit astronomers who went there.

The final item of circumstantial evidence adduced by AG is the similarity of Tycho Brahe's model of the universe to that proposed by Nilakantha in his *Tantrasangraha*.[8] In both cases the earth is considered to be at the centre of the universe but the other known planets are treated as revolving around the sun. Although there is a distinct difference not taken into account by AG – namely that Nilakantha adopted the model as a mathematically convenient computational fiction in contrast to Brahe who saw it as physically real - it is significant the two models have such close resemblance. They see this as another piece of evidence that the Kerala School influenced 17th century European mathematics.

Hence AG conclude that there is a great deal of circumstantial evidence suggesting that European mathematical astronomers learnt from their Indian counterparts after the Portuguese developed trading contacts with India in 1498. They conclude that the development of the calculus in Europe in the second century after such contacts is likely to have been facilitated by European knowledge of Indian mathematical discoveries of the Kerala School.

Nevertheless when AG start looking for concrete evidence of transmission in Jesuit communications to European mathematicians

and scientists by close examination of their correspondences in archival materials they draw a blank. This is in striking contrast to the abundant evidence available indicating the import of Indian knowledge into Europe prior to the 14th century and following late 17th century. But no evidence of documentary transmission of Indian mathematical ideas into Europe appears to exist during the time the key ideas of the calculus developed in India in the 14th century and reached their culmination in Europe in the 17th century. Since the circumstantial evidence for transmission appears strong they hope for more progress in uncovering documentary evidence in the future.[9]

3. Transmitting Skill-Based Knowledge without Documentation

However, from a methodological point of view the search for documentary evidence may be flawed. It presumes that the Indian discoveries were transmitted to Europe as a system of theoretical ideas (i.e. as propositional knowledge involving 'know-that' claims) rather than as a system of computing techniques (i.e. as skills-based knowledge involving 'know-how'). In India, the knowledge developed by the Kerala School appears to have been seen as knowledge of rules of computation. Thus, unlike the Greek and Platonic conceptions of mathematics as discovering truths about the world (or the ideal world of forms), Indians saw their mathematical rules as analogous to the rules of grammar discovered by Panini. These rules defined correct practices rather than true beliefs. Hence, they were quite prepared to see them as abstracted from experience as Panini abstracted his grammatical rules from experience, and to constantly refine these rules by strategic fine-tuning to give better computational accuracy. It was this flexible and pragmatic search for better computational techniques that led Indian mathematicians to the discovery of the Indian number system, negative numbers, infinite series representations of irrational numbers, logarithms, trigonometry and mathematical series representations of circular functions.

The Indian approach to mathematics as computation techniques suggests that AG may be looking in the wrong

direction when they seek evidence for transmission in letters, documents, translated texts etc. communicated by Jesuit missionary scientists and scholars to their counterparts in Europe. Firstly, the Jesuits would not see such knowledge as worthy of communication *qua* being mathematical knowledge. It would have appeared to them as having only utilitarian value – the kind of practical rule-of-thumb knowledge that would interest only cartographers, mariners and calendar makers. But this would not have stopped them from using their training in mathematical and the Indian languages to translate such Indian mathematical techniques for use by their own cartographers, navigators and calendar makers.

What is being suggested is that while there may have been no obstacle to the assimilation of the discoveries of the Kerala School as technical knowledge by European craftsmen, there would have been barriers to transmitting such knowledge to European scholars as a form of intellectual knowledge. It lay in the epistemological divide which separated European conceptions of mathematics from the Indian conceptions. These would have been sufficiently wide for the Jesuits *not* to see the computation techniques they were translating and teaching as a part of the high tradition of mathematics worthy of communication to scholars in Europe, any more than translations of techniques of ship-making, say. Moreover, the absence of a tradition of proof in many Indian texts, so crucial to the way mathematical knowledge was considered to be structured in Europe under the influence of Plato and Euclid, would have precluded even recognition of these discoveries as constituting a part of the mathematical tradition.

Hence, we should also be open to the possibility that the Indian mathematical discoveries may have reached Europe as a set of practical computing rules rather than a body of mathematical discoveries. Even in India the mathematical discoveries of the Kerala School were actually seen as practical empirical knowledge of techniques needed to make astronomical calculations – calculations used to determine religious holy days, design calendars that could predict likely times of arrival of the monsoons and so decide probable times for sowing and harvesting crops, and make charts for use by navigators and

mariners. Indeed, Narasimha has recently argued that the Indian approach to mathematics can be described as 'computational positivism' designed essentially to make calculations that would fit observation with increasingly precision.[10] According to him the approach is largely inspired by the algorithmic orientation of Indian grammar where there is little emphasis on theories and models, and greater regard for tuning algorithms to yield improved predictions.[11]

Moreover, the transmission of the knowledge to Europe through practical uses would also suggest that it would not have been the whole infinite series identified by Indian mathematicians (who might be interested in them for their intrinsic mathematical properties) that would have been communicated. Instead it would only have been a truncated version of these infinite series, which involved a few of the initial terms (how many depending on the degree of precision required and the rapidity of convergence of the series), that would have reached Europe. Even within India these discoveries would have been passed down to craftsmen in the form of a finite series. Since it was the custom of the elite mathematical astronomers to present their results to royal houses which supported them, it is likely that they were adapted into finite series approximations before transmission to craftsmen who needed them. These same craftsmen may also be involved in directly transmitting them to their European counterparts as they worked on Portuguese ships, and in the many Portuguese schools that were set up in India.[12]

It also makes possible a direct transmission of such know-how from their Indian counterparts to European cartographers, navigators and calendar makers without any Jesuit mediation as they worked together under various contexts. The Indian computing techniques would disseminate widely simply because they were better than those available to the Europeans. If the most significant contribution of the Kerala School was moving from the finite procedures of ancient mathematics and treating their limit passage to infinity, then for practical computation purposes it is necessary to move from the infinite series representations to a finite series approximation. The transmitted finite series could have been used by European mathematicians

to reconstruct the original infinite series discovered by Indian mathematicians.

To see how such a process could have occurred let us consider the power series for arctanx discovered by Gregory in 1667. It can be written as follows:

$\text{Arctan}^{-1}x = x - x^3/3 + x^5/5 + \ldots$ [Gregory Series for Arctan]

Jyesthadeva's *Yuktibhasa* gives the infinite series expansion for arctan as formulated by Madhava as follows:

"The first term is the product of the given Sine and radius of the desired arc divided by the cosine of the arc. The succeeding terms are obtained by a process of iteration when the first term is repeatedly multiplied by the square of the Sine and divided by the square of the Cosine. All the terms are then divided by the odd numbers 1, 3, 5, ...The arc is obtained by adding and subtracting [respectively] the terms of the odd rank and those of even rank. It is laid down that the [Sine of the] arc or that of its complement whichever is smaller should be taken here [as the given Sine]. Otherwise, the terms obtained by the above iteration will not tend to the vanishing magnitude."[13]

The above formulation of the rule for generating the infinite series for inverse tangent is given in terms of the Indian sine and cosine expressed in capital letters such that Sin θ = rSin θ and Cos θ = rCos θ. Joseph notes that "the condition given at the end of the rule may be interpreted as ensuring that rsinθ is less than rcosθ, or that tanθ (i.e. x) should be less than 1 to ensure absolute convergence of the series." Joseph then argues that the Madhava rule for the power series for inverse tangent may be written as follows using modern notation:

$$r\theta = r(r\sin\theta)/1(r\cos\theta) - r(r\sin\theta)^3/3(r\cos\theta)^3 + r(r\sin\theta)^5/5r(\cos\theta)^5 - \ldots$$

Therefore:

$$\theta = \tan\theta - \tan^3\theta/3 + \tan^5\theta/5 - \ldots \quad \text{[Madhava Series for Arctan]}$$

Joseph notes that the above formula is equivalent to the Gregory series for inverse tangent.

It is important to note that the rules for generating the infinite series are given by Madhva as iterative rules. This allows the series to be prolonged indefinitely to any number of terms we wish by repeated application of the rules. For values of $x < 1$ the series converges since succeeding terms become smaller and smaller. Hence for practical purposes we may take only three terms, say, to produce an arctan table for practical use by craftsmen. Moreover, the close similarity in Madhva and Gregory series, which would have been identical if not for the fact that the Indian Sine and Western sine are different since Sinx = rsinx, makes it easy to see how knowledge of the use of the Madhva series approximations by Indian craftsmen could have been adapted to the Western case. The new approximate series would arrive in Europe as a computing technique used by craftsmen. It could serve as inspiration and stepping stone for European mathematicians to reconstruct infinite series expansion for arctanx equivalent to those already known to their Indian counterparts. They would, of course, look for ways of proving it so that it would become a part of mathematics as they saw it.

Nevertheless, in the process of rediscovering the rule European mathematicians would not be aware of the contributions of their Indian counterparts to their own accomplishments. The process of rediscovery would have precluded such recognition. Firstly, they did not receive the rule for the infinite series directly but reconstructed it from a finite series used as a practical rule-of-thumb by European craftsmen. Secondly, the finite series they reconstructed had itself been modified to fit Western trigonometric functions which were slightly different from the Indian equivalents. Finally, they would have been prepared to acknowledge discovery of an infinite series expansion as mathematically significant only after demonstrative proof for it [even if there was controversy about the adequacy of the proof]. Hence, on all of these grounds, the Indian influence on them would have been invisible to European mathematicians.

This would also explain why European thinkers tended to trace the new ideas to the ancient Greeks. In particular the influence of the method of exhaustion developed by Leucippus, Democritus and Antiphon is cited. This method was put on a

more rigorous basis by Eudoxus, and is seen as culminating in the brilliant achievements of Archimedes. He used it to calculate areas and volumes of figures by summing up more and more areas as part of an infinite series. His use of the method of exhaustion to find an approximation for the area of a circle also led to an approximation for π more accurate than anything known earlier. Archimedes used it to compute the area of an ellipse, the volumes and surface areas of spheres and cones, and the volumes of any segment of revolution of a paraboloid or hyperboloid of revolution using the method as form of 'integration'. Given that these achievements of the ancient Greeks were the closest approaches to the infinite series and calculus they discovered, European mathematicians easily saw them as the only pioneering achievements upon which they stood. Indeed, Luca Valerio's (1552-1618) *De quadratura parabolae* published in Rome in 1606 saw itself as continuing this Greek tradition of attacking problems in mathematics through the method of exhaustion.

However, there are reasons to suspect that this historical consciousness is wrong, given the developments in both Indian and European mathematics in the 16^{th} and 17^{th} centuries. They involved radical epistemological and methodological breaks that are better explained as the outcome of a dialogical exchange between the two traditions. Consider the Indian case first. It is significant that after the arrival of the Portuguese in India the discoveries of the Kerala School came to be presented in an entirely different way. The new mode of presentation involved a break in both language and epistemological orientation. The change is evident in the work of Jyesthadeva. He was born in 1500 shortly after the arrival of the Portuguese, and in the same area where Portuguese were to set up Jesuit inspired schools while he was growing up. His *Yuktibhasa*, written about 1550, summarizes all the previous accomplishments of the Kerala School, and is a unique text for two reasons. Firstly it ruptures with tradition by being written in Malayalam, the regional language of Kerala, and not in Sanskrit – the traditional language used by his predecessors and still in use by the intellectual elites where he lived. Secondly, unlike his predecessors, Jyesthadeva gives detailed proofs of theorems and derivations of rules he describes in his study.

This dramatic reorientation of the language and epistemology of the *Yuktibhāsa* raises the question of who could have been his intended audience. Firstly it is possible to suspect that it could not be an audience conversant with Sanskrit since it would not have been worth the effort of translating ideas from one linguistic medium to another. Secondly, the audience could not have been satisfied with mere descriptions of the discoveries of the school - it also demanded proof. Hence it cannot be geared to serve mere craftsmen who would have been more interested in applications than proofs.

I suggest that the work is designed for those who were also studying astronomy, and techniques of calendar making and navigation, in the schools that the Portuguese had set up. These practitioners would no longer be willing to accept the results of the Kerala School at face value to be handed down as inspired by authority. Hence, the *Yuktibhasa* could have been intended for an audience with no knowledge of Sanskrit, but were educated enough to demand proof of theorems. In the past the Indian mathematicians may have taught the proofs to their disciples, but did not include it in the texts in which their discoveries came to be disseminated more widely to craftsmen in general. It now became necessary to include proofs, and in the vernacular language too, given the competition introduced by the alternative astronomical tradition introduced by the Portuguese.

It is also noteworthy that even within the European tradition there were changes after direct contact between India and Europe. Europeans began to approach problems associated with infinite sums in a way that resembled more the method direct rectification used by the Indians rather than the method of exhaustion used by the ancient Greeks. Joseph describes the difference by using the example of the derivation of the arctan Madhva series given in the *Yuktibhasa*:

"The approach involves what is known as the direct rectification of an arc of a circle, i.e. the summation of very small arc segments and reducing the resulting sum to an integral. ...This is a very interesting geometric technique different from the method of exhaustion used in the Arab and European

mathematics. In the Kerala case you are sub-dividing an arc into *unequal* parts and while in the other (Arab and European) case there is a sub-division of the arc into *equal* parts. The different technique used in Kerala was not because the method of exhaustion was unknown to the Indians. Indeed, it is likely that Aryabhata [in the 5^{th} century] used the method of exhaustion to arrive at his accurate estimate of the circumference for a given diameter. The 'exhaustion' method was probably avoided because the calculation involved working out the square roots of numbers at each stage of the calculation, a tedious and time-consuming task."[14]

This Indian method of summing mathematical series and computing areas, using the method of rectification, can be seen as growing out of the geometrical approach to representing numbers developed by Nilakantha in solving problems in arithmetical progressions.[15] It is used by Jyesthadeva in the *Yuktibhasa* to derive the Madhava-Gregory inverse tangent series. This derivation not only shows how trigonometric functions can be expressed in the form of an infinite series but, according to Ramachandran who examined the derivation in detail, also "involves several techniques including the idea of integration and differentiation".[16] Indeed the notion of integration and differentiation in the calculus is intimately linked with the notion of the sum of a series and the limit term of a series. Hence it seems reasonable to conclude that the Kerala School's achievements were much closer to modern calculus than the method of exhaustion that is so often treated as its direct predecessor.

The changes of approach in mathematics made by both Jyesthadeva and European mathematicians suggest contact between Indian and European traditions of mathematics in the early modern era. This would also explain why many of the major new developments in the emergence of modern calculus were taken by mathematicians born within a few years of each other. Fermat, Roberval and Cavalieri were born fifty years after Jyesthadeva's *Yuktibhasa* was published in Malayalam, and within five years of each other at the turn of the 16^{th} century.

If the arguments above are taken as reasonable it is quite possible that the Indian computation techniques developed by the

Kerala School may have reached Europe as a body of approximate series for various circular functions, reconstructed as part of an infinite series by European mathematicians, and become historically explained as extensions of the method of exhaustion used by ancient Greek mathematicians. The fact that they had a rational basis in Indian mathematics would neither have been known to those who carried the approximate series into Europe, nor to those who reconstructed their infinite complete forms in the new culture.[17]

4. Establishing Transmission without Documentary Evidence

The transmission of the discoveries of Kerala mathematics as know-how and computation techniques through the channel of craftsmen and technicians would explain the absence of documentary evidence in Jesuit communications. Even if there is documentary evidence in 16^{th} century European manuals used for navigation, map-making and calendar construction of the use of approximate series derived from the discoveries of the Kerala School, it would hardly have been directly communicated to European mathematicians. After all craftsmen oriented to practical rather than theoretical concerns are not likely to write to leading mathematical figures in Europe – or be taken seriously if they did so.

Moreover, even if we were to find European ship records, say, that show the use of approximate series corresponding to their Indian infinite series counterparts it would not establish transmission. There is still the possibility of independent European discovery of these approximate series. Indeed without documentary evidence in the form of direct translations of Indian texts, or direct acknowledgment of Indian sources for these discoveries, the legal criterion carries a crucial weakness when it is extended to apply to the context of establishing scientific and mathematical discoveries. Since these are discoveries concerning an objective world, or the best techniques for solving problems, it is always possible for different people to make the same discoveries. Hence the similarity in the discoveries made need not necessarily imply influence – it may simply be the outcome of the independent discoveries of the best description of the

world or best way of solving a problem. Since what works does not depend merely on our arbitrary choices, but is dictated by objective constraints, we might even expect this to occur. For example, many cultures have independently discovered the right-angle theorem attributed to Pythagoras in the West, but this is not the outcome of diffusion of knowledge but objective states of affairs. They were independent discoveries that yielded the same knowledge needed for constructing buildings in all cultures.

Furthermore, when we apply the legal criterion, where motivation is a factor, its strong presence could also be invoked to make a case for independent discovery. Since the Europeans were motivated to discovering the requisite knowledge they were led to discover it independently. Hence, without additional documentary evidence it may appear that the presence of the motivational factor may even weaken whatever circumstantial evidence is discovered in the form of similarities in discoveries made. Hence it becomes crucial to ask: Can the application of the legal criterion be strengthened to apply to cases of transmission of know-how (practices and techniques) where there is no documentary evidence to buttress our case?

An approach to answering this question can be made by looking at a parallel case of transmission of Indian mathematical discoveries; the Indian number system. How was the evidence for the discovery of the number system and its diffusion into Europe established? One key argument has been the similarities in the number system that came to be used by Europeans with earlier used in India. However, this would not have clinched the case for transmission because it could always be argued that the Europeans could have independently discovered the number decimal-place number system with zero. After all the use of decimal numbers was already well-established in both regions very early in their history, and the place-value system as known to the Babylonians, and the invention of zero had also been made by the Mayans independent of the Indians. Hence, nothing precluded the possibility of a parallel discovery by the Europeans and Indians of the currently dominant number system. However, the evidence that finally became convincing was the discovery of accidental similarities in the number systems used by Europeans and Indians – namely that the signs for numbers

used in Europe were similar to the signs used earlier in India. Since the choice of signs is conventional, and the same number system could have been represented by different number signs, it became necessary to assume that the parallels could only be explained by transmission. It is such accidental similarities, conjoined to the crucial similarities that make the number system function, that furnish the strong circumstantial evidence for transmission.

According to Menninger, Indian numerals came to be used in Alexandria sometime in the 5th century A.D. before penetrating into Europe. However, they did not arrive in Europe as part of theoretical mathematical inquiries, or via scientific treatises, but as skills and practices that came to be known in harbors and ports.[18] They were transmitted as computing skills – precisely the way we are suggesting the discoveries in infinite series also came to be transmitted. Yet it was possible to establish that they came from India because the number signs used in Europe were quite similar to those used earlier in India. Such accidental parallels cannot be easily explained by assuming that they were independent achievements in the same way, say, the discovery of a decimal place system with zero in Europe could have been. The intimate association of these accidental similarities with the essential similarities of the number system establishes its origins in India and transmission to Europe. This suggests that in order to establish the transmission of infinite series representations of trigonometric functions from India to Europe we could look for accidental correlations, over and above essential similarities, in the infinite series representations in Europe and India – similarities that cannot be explained away easily as independently discovered parallels.[19]

4. Implications for Future Research

In this context I would like to raise the following question: Were the earliest infinite series for π, sinx, cosx, and $\tan^{-1}x$ discovered in Europe identical to the series discovered in India? Given that there are many possible infinite series expansion of π and the various trigonometric functions would we have to describe the similarity in the series discovered as a miraculous coincidence if

we refuse to accept diffusion? In particular, is it more reasonable to suppose that it was the approximate series expansions of π and various trigonometric functions discovered in India that inspired the first infinite series expansions series of them in Europe? If so would it not be reasonable to suppose that the Indian discoveries inspired the European ones and that, if Europeans had made their discoveries independently they are hardly likely to have arrived in the beginning at the same infinite series expansions as the Indians given there are potentially a large number of expansions of these same functions and pi?

DISCUSSION

The central hypothesis of this paper was that if there was transmission of knowledge of infinite series to Europe, it was done indirectly through practical uses, with a truncated version being passed on from local craftsmen to their foreign counterparts (such as navigators) and then being reconstructed in Europe by the mathematically knowledgeable without being aware of its provenance. The question was raised whether it was possible to transmit the knowledge of higher mathematics like the infinite series, through computations and calculations contained in navigation charts and similar aids, and if so what would be the precise nature of the calculation with series that could be transmitted. Within Europe of that time, there had already been established a tradition of tables that received a considerable boost with the establishment of printing presses that had assured the availability of these tables to a wider population. Only a closer look at ships records and other practical manuals would help to resolve the validity of this hypothesis.

The question was also raised as to how necessary was it to obtain increasingly accurate values for sine, cosine and π from a practical point of view. Opinions were divided as to whether accuracy (up to 10 or more decimal places) were required for navigational purposes. Of course, a 'delight in accurate calculation' may have driven the Kerala mathematicians to attempt increasingly accurate approximations.

The whole issue of internal transmissions as a prelude to crossing the boundaries is of central importance and was

discussed. It was felt that there was a need to look at the background of people involved, not just of patrons from the royal courts, but also members of *ahargan* who were more likely to be in closer contact with practitioners on matters of a commercial or navigational nature. Evidence had to be sought of the Portuguese setting up schools where the medium of instruction was Malayalam or where there was significant intellectual contacts between the local people and the Jesuits. This was not so in the case of Goa where the schools were conducted in only Portuguese. Evidence of Jesuit mastery of local languages, of Jesuit translations into Malayalam of texts of scriptures, or training of local people to teach in schools would also be pertinent information.

In the course of the discussion, it was pointed out that if the hypothesis of transmission through navigational charts and cartographical calculations is to be sustained, one should take account of the fact that the Portuguese navigators who reached Kerala at that particular time possessed less expertise in these aspects than the North-Western Europeans who excelled in map-making etc. This raised a problem in historical logistics. However, in the case of the Jesuits, they were recruited from all over Europe, with a disproportionate number originating from Central and North West Europe.

It was pointed out during the discussion that an emphasis on non-textual transmission raised the question of the distinction between practical mathematics and higher mathematics since, by its very nature, oral transmission involved a lower level of mathematics. However, there was disagreement as to whether such a distinction was valid and what was the level of mathematics required for an exercise involved in getting greater accuracy of tables using partial corrections and other approximation procedures.

It was agreed that the question of transmission hinged to a degree on the nature of socio-cultural set-up of Kerala at that particular time. The general assumption is that the Brahminical class contributed to the study of higher mathematics. Was it possible that the other classes of society apart from the Brahmins, like the *Ganaka*, may have passed on the knowledge in vernacular language to the Portuguese? The nature and extent

of the symbiotic relationship between the Nambuthiri Brahmin and the Ganaka in the practice of astrology and medicine would support this point. This is worth further investigation and is pertinent to the conclusions of the paper.

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ENDNOTES

1 See A Aaboe and J L Berggren, 'Didactical and other remarks on some theorems of Archimedes and infinitesimals', *Centaurus* 38 (4) (1996), 295-316. The paper argues that Archimedes *may* have used infinitesimals to produce his results.

2 Moreover, it would not have been possible for the ancient Greeks to make these discoveries since the science of trigonometry which it presupposes was not advanced among the Greeks, and had to be systematically developed in India for nearly a millennium before the discovery of the infinite series expansions of trigonometric functions by Madhava in the 14th century.

3 Indeed in the past there has been no hesitation in claiming influence whenever an idea was seen as having originated in Europe before emerging elsewhere to claim actual transmission based on possibility of contact. However, there has also been a tendency among European historians to assume that the fact that an idea had emerged outside Europe earlier was not sufficient to establish transmission even with actual contact by arguing that nothing precluded it from being an independent discovery in Europe.

4 According to AG, Galileo competed in vain for the Spanish prize for 16 years before shifting his attention to the Dutch prize. They also refer to Colbert who wrote to leading scientists in Europe, and started the French Royal Academy, to improve maps, sailing charts and the science of navigation. Prior to the formation of the British Royal Society one of the major concerns of scientists in London meeting together was the problem of longitudes. Hence, there is no disputing the motivation of European scientists in acquiring

knowledge of computing techniques that would improve navigation, calendars and practical mathematics from whatever source was available.

5 Christopher Clavius had revised the Jesuit curriculum in the 1570's at the Collegio Romano so that when the famous Matteo Ricci studied under Clavius he was also trained in cosmography and nautical science. Ricci was to use his knowledge of map-making, clock-making and astronomy later in China to win a position in the highest circles of the Chinese court and its astronomical elites. However, prior to his mission to China, the contact with India and Ricci's mathematical training, and his communication with the Jesuit Collegio Romano and Clavius, provided ample opportunity for acquiring the Indian mathematical knowledge that would have served the demand in Europe for better mathematical techniques for navigation and calendar making.

6 Aryabhata Group, p. 41

7 According to the Aryabhata Group many of the methods they adopted had no epistemological basis in European mathematics – worse it went counter to the epistemological orientation they adopted. Firstly, European mathematics was orientated towards the 'proof' tradition rather than the 'calculation' tradition of Indian mathematics. Hence, without any proof the formula would not be seen as a part of mathematics. Yet it was only two centuries later that European mathematics was to create a foundation for the formula within its proof tradition. Secondly Europeans shared the Greek 'horror of the infinite' in mathematics as in physics and cosmology. Worse Christian theology held that the property of infinity could only be attributed to God.

8 We could also add here the fact that "a key development of pre-calculus Europe, that of generalisation on the basis of induction, has deep methodological similarities with the corresponding Kerala development (200 years before). There is further evidence that John Wallis (1665) gave a recurrence relation and proof of Pythagoras theorem exactly as Bhaskara II did. The only way European scholars at this time could have been aware of the work of Bhaskara would have been through Keralese 'routes'". See http://www-groups.dcs.st-and.ac.uk/~history/Projects/Pearce/Chapters/Ch9_4.html [Last accessed 21 February 2006]

9 Aryabhata Group, pp. 47-48.

10 Narasimha (2003) p.60

11 Narasimha (2003) pp. 60-63

[12] In a personal communication to the author Rajasekhar Parameswara Aiyer confirmed such a possibility by writing:

"Madhava's thumb rules for finding Sine Chords (*Jya*) , Rule containing first few terms of series expansions for RSine (Jya), Rcos (*Kotijya*), etc were orally transmitted from generation to generation. These rules are abundantly quoted. In *AryaBhateeya Bhashya* (Interpretation of AryaBhateeyam) by Nilakanta, *Yukti Bhasa* of Jyeshtadeva , *Kriyakramakarai* of Sankara etc we can find these. (Many other such rules existed for different computations) As the name of the book indicates, "Yuktibhasha" gives the derivation of these rules explaining the rationale behind each rule. These rules are like ready made tables, a user who doesn't have much in depth knowledge can also use it, to find the desired result , by blindly carrying the computation as instructed. These computations were done with out paper (writing) and *"Kavidi Kriya"* (computation with the help of Kavidi (Conch shells) and only final results were noted down in palm leaf. Even now this method is not extinct in Kerala and some traditional Astronomers are using it. The rules cited above were mainly used by Astronomers and those who were involved in *Panchanga Ganana* (Calendar making)."

[13] See Joseph (2000) p.290.

[14] Joseph (2002) p. 131

[15] See Mallayya (2002) for more detailed discussion.

[16] Ramachandran (2002) p. 138

[17] Moreover, the Portuguese state saw all knowledge of maps and map-making, as well as navigation and calendar techniques as secrets of state that were prohibited from being widely disseminated. This would also have precluded Jesuits from documenting this knowledge even in the letters sent back to Europe which and intended for wide circulation. Hence, even as the Jesuits participated in communicating the know-how of Indian techniques they would not transmit this knowledge in documented form back to Europe.

[18] See Menninger (1970) p. 406. See also Gupta (1983) p.23.

[19] Indeed many other transmissions also took place without leaving documentary evidence. Consider the case of the origins of gunpowder, printing and compass which puzzled Bacon near the dawn of the modern era:

It is well to observe the force and virtue and consequences of discoveries. They are to be seen nowhere more conspicuously that

in those three which were unknown to the ancients, and of which the origin, though recent, is obscure and inglorious; namely printing, gunpowder, and the magnet. For these three have changed the whole face and state of things throughout the world, the first in literature, the second in warfare, the third in navigation; whence have followed innumerable changes; insomuch that no empire, no sect, no star, seems to have exerted greater power and influence in human affairs than these mechanical discoveries. [Quoted in Needham (1954), p.19]

We now know how these transmissions were originated in China and came to Europe largely through the corridor of communication created by the Mongolian empire.

8

THE PHILOSOPHY OF MATHEMATICS, VALUES, AND KERALESE MATHEMATICS

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WHAT IS THE BUSINESS OF THE PHILOSOPHY OF MATHEMATICS?

Traditionally, in Western philosophy, mathematical knowledge has been understood as universal and absolute knowledge, whose epistemological status sets it above all other forms of knowledge. The traditional foundationalist schools of formalism, logicism and intuitionism sought to establish the absolute validity of mathematical knowledge by erecting foundational systems. Although modern philosophy of mathematics has in part moved away from this dogma of absolutism, it is still very influential, and needs to be critiqued. So I wish to begin by summarising some of the arguments against Absolutism, as this position has been termed (Ernest 1991, 1998).

My argument is that the claim of the absolute validity for mathematical knowledge cannot be sustained. The primary basis for this claim is that mathematical knowledge rests on certain and necessary proofs. But proof in mathematics assumes the truth, correctness, or consistency of an underlying axiom set, and of logical rules and axioms or postulates. The truth of this basis cannot be established on pain of creating a vicious circle

(Lakatos 1962). Overall the correctness or consistency of mathematical theories and truths cannot be established in non-trivial cases. (Gödel 1931).

Thus mathematical proof can be taken as absolutely correct only if certain unjustified assumptions made. First, it must be assumed that absolute standards of rigour are attained. But there are no grounds for assuming this (Tymoczko 1986). Second, it must be assumed that any proof can be made perfectly rigorous. But virtually all accepted mathematical proofs are informal proofs, and there are no grounds for assuming that such a transformation can be made (Lakatos 1978). Third, it must be assumed that the checking of rigorous proofs for correctness is possible. But checking is already deeply problematic, and the further formalizing of informal proofs will lengthen them and make checking practically impossible. (MacKenzie 1993)

A final but inescapably telling argument will suffice to show that absolute rigour is an unattainable ideal. The argument is well-known. Mathematical proof as an epistemological warrant depends on the assumed safety of axiomatic systems and proof in mathematics. But Gödel's (1931) second incompleteness theorem means that consistency and hence establishing the correctness and safety of mathematical systems is indemonstrable. We can never be sure mathematics theories are safe, and hence we cannot claim their correctness, let alone their necessity or certainty. These arguments are necessarily compressed here, but are treated fully elsewhere (e.g., Ernest 1991, 1998). So the claim of absolute validity for mathematical knowledge is unjustified.

The past two decades has seen a growing acceptance of the weakness of absolutist accounts of mathematical knowledge and of the impossibility in establishing knowledge claims absolutely. In particular the 'maverick' tradition, to use Kitcher and Aspray's (1988) phrase, in the philosophy of mathematics questions the absolute status of mathematical knowledge and suggest that a reconceptualisation of philosophy of mathematics is needed (Davis and Hersh 1980, Lakatos 1976, Tymoczko 1986, Kitcher 1984, Ernest 1997). The main claim of the 'maverick' tradition is that mathematical knowledge is fallible. In addition, the narrow academic focus of the philosophy of

mathematics on foundationist epistemology or on Platonistic ontology to the exclusion of the history and practice of mathematics, is viewed by many as misguided, and by some as damaging.

RECONCEPTUALIZING THE PHILOSOPHY OF MATHEMATICS

Although a widespread goal of traditional philosophies of mathematics is to reconstruct mathematics in a vain foundationalist quest for certainty, but a number of philosophers of mathematics agree this goal is inappropriate. "To confuse description and programme - to confuse 'is' with 'ought to be' or 'should be' - is just as harmful in the philosophy of mathematics as elsewhere." (Körner 1960: 12), and "the job of the philosopher of mathematics is to describe and explain mathematics, not to reform it." (Maddy 1990: 28). Lakatos, in a characteristically witty and forceful way which paraphrases Kant indicates the direction that a reconceptualised philosophy of mathematics should follow. "The history of mathematics, lacking the guidance of philosophy has become blind, while the philosophy of mathematics turning its back on the...history of mathematics, has become empty" (1976: 2).

Building on these and other suggestions it might be expected that an adequate philosophy of mathematics should account for a number of aspects of mathematics including the following:

1. **Epistemology**: Mathematical knowledge; its character, genesis and justification, with special attention to the role of proof
2. **Theories**: Mathematical theories, both constructive and structural: their character and development, and issues of appraisal and evaluation
3. **Ontology**: The objects of mathematics: their character, origins and relationship with the language of mathematics, the issue of Platonism
4. **Methodology and History:** Mathematical practice: its character, and the mathematical activities of mathematicians, in the present and past

5. **Applications and Values**: Applications of mathematics; its relationship with science, technology, other areas of knowledge and values
6. **Individual Knowledge and Learning**: The learning of mathematics: its character and role in the onward transmission of mathematical knowledge, and in the creativity of individual mathematicians (Ernest 1998)

Items 1 and 3 include the traditional epistemological and ontological focuses of the philosophy of mathematics, broadened to add a concern with the genesis of mathematical knowledge and objects of mathematics, as well as with language. Item 2 adds a concern with the form that mathematical knowledge usually takes: mathematical theories. Items 4 and 5 go beyond the traditional boundaries by admitting the applications of mathematics and human mathematical practice as legitimate philosophical concerns, as well as its relations with other areas of human knowledge and values. Item 6 adds a concern with how mathematics is transmitted onwards from one generation to the next, and in particular, how it is learnt by individuals, and the dialectical relation between individuals and existing knowledge in creativity.

The legitimacy of these extended concerns arises from the need to consider the relationship between mathematics and its corporeal agents, i.e., human beings. They are required to accommodate what on the face of it is the simple and clear task of the philosophy of mathematics, namely to give an account of mathematics.

CHALLENGING EPISTEMOLOGICAL ASSUMPTION AND VALUES

The challenge to the traditional philosophy of mathematics to broaden its epistemological goal, as indicated above, raises some critical issues. In particular, if providing ironclad foundations to mathematical knowledge and mathematical truth is not the main purpose of philosophy of mathematics, has this fixation distorted philosophical accounts of mathematics and what is deemed valuable or significant in mathematics? To what extent is the philosophical emphasis on mathematical proof and deductive theories justified? I want to argue that the emphasis on mathematics as made up of rigorous deductive theories is excessive, and this focus in fact existed for only two periods totaling possibly less than ten percent of the overall history of mathematics as a systematic discipline, and then only in the West.[1]

The first of these two periods was the ancient Greek phase in the history of mathematics which reached its high point in the formulation of Euclid's *Elements*, a systematic exposition of deductive geometry and other topics. The second period is the modern era encompassing the past two hundred years or so. This second period was first signaled by Descartes' modernist epistemology, with its call to systematize all knowledge after the model of geometry in Euclid's *Elements*. However, fortunately, his injunction was not applied in the practices of mathematicians for the next two hundred years, which was instead a period of great creativity and invention in the West. Only in the 19th century did the newly professionalized mathematicians turn their attention to the foundations of mathematical knowledge and systematize it into axiomatic mathematical theories. The contributions of Boole, Weierstrass, Dedekind, Cantor, Peano, Hilbert, Frege, Russell and others in this enterprise up to the time of Bourbaki are well known.

I am not claiming that all or even most mathematical work was foundational during these two exceptional periods. But the foundational work is what caught the attention of philosophers of mathematics, and in the spirit of Cartesian modernism has become the epistemological focus of modern philosophy of mathematics, as well as the touchstone for what is deemed to be of epistemologically valuable. I do not want to detract from either the magnificence of the achievement in the foundational work carried out by mathematicians and logicians, nor from the pressing nature of the problems that made attention to it so vital in the early part of the 20th century. Nevertheless, the legacy of this attention has been to overvalue the philosophical significance of axiomatic mathematics at the expense of other dimensions of mathematics. Two underemphasized dimensions of mathematics are calculation and problem solving. All three of

these aspects of mathematics involve deductive reasoning, but axiomatic mathematics is valued above the others as the supreme achievement of mathematics.

There is another feature shared by the two historical periods that emphasised axiomatic mathematics, namely a purist ideology involving the philosophical dismissal or rejection of the significance of practical mathematics. The antipathy of the ancient Greek philosophers to practical matters including numeration and calculation is well known. This aspect of mathematics was termed 'logistic' and regarded as the business of slaves or lesser beings. In the modern era, calculation and practical mathematics have been viewed as mathematically trivial and philosophically uninteresting. The fact that philosophers have been concerned with ontology and the nature of the mathematical objects has engendered little or no interest in the symbolism of mathematics, or calculations and transformations that convert one mathematical object (or rather its name, a term) into another. Such a view is typified by Platonism, which concerns itself primarily with mathematical truths and objects. These are presumed to exist in an unearthly and idealized world beyond that which we inhabit as fleshy and social human beings, such as Popper's (1979) objective World 3.

Of course at the same time as these modern developments were taking place applied mathematics and theoretical or mathematical physics were making great strides, but this was not considered to be of interest to philosophers of mathematics (however much interest it was to philosophers of science), because of their purist ideology. Even in British public schools, during the late Victorian era, mathematics was taught in with ungraduated rulers because graduations implied measurement and practical applications, which was looked down upon for the future professional classes and rulers of the country. (Admittedly some of the rationale was that Euclid's geometry only requires a straight-edge and a pair of compasses as drawing instruments).

What I have described here (in order to critique it) is an ideological perspective that elevates some aspects of mathematics above others, but typically does not acknowledge that it is based on a set of values, a set of choices and preferences to which no necessity or logical compulsion is attached.

Furthermore, it appears that such values have only been prominent during a small part of the history of mathematics.

In order to strengthen my critique of these values I want to point out that mathematical proof, the cornerstone of axiomatic mathematics, and calculation in mathematics, are formally very close in structure and character. In Ernest (forthcoming) I have argued that mathematical topic areas (e.g., number and calculation) can be interpreted as being made up semiotic systems, each comprising (1) a set of signs, (2) rules of sign production and transformation, and (3) an underpinning (informal) meaning structure. Such signs include atomic, i.e., basic, signs and a range of composite signs comprising molecular constellations of atomic signs. These signs may be alphanumeric (made up of numerals or letters) or figural (e.g., geometric figures) or include both (e.g., figures with labels and the types of inference employed). The use of semiotic systems is primarily that of sign production in the pursuit of some goal (e.g., solving a problem, making a calculation, producing a proof for a theorem). I want to claim that most recorded mathematical activity concerns the production of sequences of signs (within a semiotic system). Typically these are transformations of an initial composite sign (S_1), resulting after a finite number (n) of transformations, in a terminal sign (S_{n+1}), satisfying the requirements of the activity. This can be represented by the sequence: $S_1 \rightarrow S_2 \rightarrow S_3 \rightarrow \dots \rightarrow S_{n+1}$. Each transformation (represented by $\rightarrow$) constitutes the application of one of the rules of the semiotic system to the sign, resulting in the derivation of the next sign in the sequence. More accurately these should be represented by $\rightarrow_i$, with i = 1, …, n, since each transformation in the sequence is potentially different.

My claim is that this formal (semiotic) system describes most mathematical domains and activities. If the initial sign is the statement of a problem, the sequence represents the derivation of a solution to the problem. I will not dwell on this case as there are many complications involved in problem solving, such as the use of multiple representations, branching solution attempts[2], etc. and some of the transformations (such as interpreting an initial problem formulation and constructing a

problem representation) are neither easily made explicit nor fully formalizable. Furthermore, there is no simple characterization of the relationship between the transformational rules and the underlying informal meaning structure, for the transformations are partly structure preserving morphisms, and partly calculational.

More significantly in the present context, such transformational sequences can represent a deductive proof for a theorem. In this case it consists of a sequence of sentences, each of which is derived from its predecessors by the deductive rules of the system (including the introduction of axioms or other assumptions). The final sign in the sequence is the theorem proved. The meaning structure underpinning the rules of proof is based on the principle of the preservation of the truth value of sentences in each deductive step, and hence along the length of the proof sequence (which is why axioms can be inserted, and why proofs 'work', i.e., do what they are designed to do.)[3]

In the case of a calculation, the initial sign is usually a compound term. Subsequent terms are derived by calculational rules and typically each is a simplification in some sense of its predecessor. The final term in a calculation is the simplified numerical 'answer' to the problem. With the introduction of algebra and other functions and operations such as trigonometrical functions, the answer may instead be a simplified but non-numerical term (i.e., a function). Thus calculations are sequences of terms, each derived from predecessors by the rules of the system. The meaning structure underpinning the rules of calculation is based on the principle of the preservation of numerical value.[4]

Thus there is a strong analogy between the semiotic systems based on calculation and those based on deductive proof. The transformations of terms and sentences are based on the underlying principles of value preservation, namely numerical value or truth value, respectively, as I have demonstrated. In addition, calculation concerns terms and proof concerns sentences (or formulas), and both of these are defined similarly. Terms (sentences) are defined recursively as follows. An atomic term (sentence) consists of a constant or a variable (an *n*-place predicate applied to constants or variables, respectively). A compound term

(sentence) is defined as the result of applying a function or operation (a logical connective or quantifier, respectively) to one or more terms (sentences, respectively), to make a new term (sentence, respectively). Thus structurally terms and sentences are very similar, defined analogously by induction.

The sequential and rule-based nature of calculation is something that precedes the development of the deductive proof of theorems by at least two thousand years. My contention is that without the long and ancient tradition of rule following in sequences of calculations, and the entrenchment of the grammatical and value preservational features noted above, the development of proof would not be possible. As I have indicated here, there is a striking analogy between calculations and deductive proofs of theorems, rarely if ever remarked upon, that puts into question the claimed superiority of proof.

Furthermore, proof and calculation are formally equivalent, in modern foundational terms. Calculations utilize the term as a basic unit of meaning (and as that which is transformed), whereas deductive proofs use the sentence (including formulas or open sentences) as a basic unit. However, there are equivalence transformations between calculations and proofs. A calculation sequence of the form $t_1, t_2, t_3, \ldots, t_n$, where each t_i $(1 \leq i \leq n)$ is a term, can be represented as a deductive proof of the form $t_1=t_2, t_2=t_3, t_3=t_4, \ldots, t_{n-1}=t_n$ in which each identity asserts that numerical values of adjacent terms are preserved identically in the calculation. By an extended or repeated application of the transitivity of identity ($x=y$ & $y=z \rightarrow x=z$, for all x, y & z), $t_1=t_n$ is derived, thus equating the initial term of the calculation and the terminal term, the 'answer'.

Likewise, a deductive proof of the form $S_1, S_2, S_3, \ldots, S_n$, can be represented as a series of terms. These are the values of the truth value function *f* defined on numerical representations of true and false sentences to give the values 1 and 0, respectively. For a valid proof these values must be $f(S_1 \rightarrow S_2) = f(S_2 \rightarrow S_3) = \ldots = f(S_{n-1} \rightarrow S_n) = 1$. The formal details are messy and omitted here (see Gödel 1931 for the introduction of arithmetization of logic, and Kleene 1952) but the principle is both simple and sound. It is well known that *f* is a morphism mapping $\langle S, \rightarrow \rangle$ onto a Boolean algebra $\langle f(S), \leq \rangle$.[5]

The very strong analogy and structural similarities between proof and calculation, including their inter-convertibility, challenges the preconception often manifested in philosophical and historical accounts of mathematics that proof is somehow intellectually superior to calculation in mathematics.[6] Looking in detail at the technical and structural aspects of proof and calculation reveals that they cannot be so easily attributed to different epistemological domains as is often claimed. It is not defensible to say that proof alone in mathematics pertains to the true, good, beautiful, to wisdom, 'high-mindedness' and the transcendent dimensions of being, and that calculation is only technical and mechanical, pertaining to the utilitarian, practical, applied, and mundane; the lowly dimensions of existence. Such assertions are part of an ideological position incorporating a set of values that overvalues pure proof-based mathematics as having epistemological significance, and undervalues calculation and applied mathematics as having only practical significance; going back to the social divisions of ancient Greek society, as noted above, and the prejudices and ideology to which it gave rise.

This preconception or prejudice is used as the basis for asserting that the contributions of some cultures and civilizations are intellectually superior to others in history of mathematics. It also undervalues the solving of problems, calculations and other local applications of deduction in mathematics (including proof, see Joseph 1994). Thus the mathematics of ancient Egypt, Mesopotamia and India, as well as other countries outside of the Greco-European tradition, is viewed as inferior and immature. Part of the argument is that only cultures that produce axiomatic proof in mathematics achieve the highest levels of abstract intellectual achievement.

I have argued that philosophical dispositions and values have underpinned a prejudice against ascribing value to certain forms of mathematical activity. In particular, that axiomatic systems are greatly valued over less systematic forms of deduction including problem solving, calculation and unsystematized proofs. Furthermore, this prejudice also maintains the contrast between and overvalues any form of proof, including unsystematized and unaxiomatized proofs over any form of calculation or problem solving.

EUROCENTRISM IN THE HISTORY AND PHILOSOPHY OF MATHEMATICS

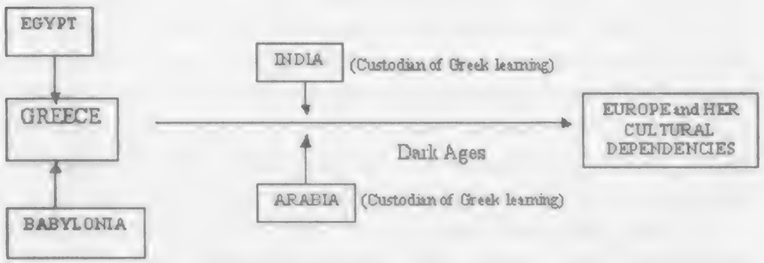
The above discussion raises the question of why informal and unsystematized proofs and demonstrations that occur in the mathematical histories of certain cultures are valued more than those of others. Why, for example, are the unsystematized proofs, methods and results of post-Renaissance European mathematics regarded as superior to antecedent developments in Kerala of comparable character? To answer this it is necessary to turn to another dimension of ideological prejudice at work in the history and philosophy of mathematics. This is eurocentrism, the racist bias that claims that the European 'mind' and its cultural products are superior to those of other peoples and races. Thus Bernal (1987) has argued that during the past two hundred year or so, ancient Greece has been 'talked up' as the starting point of modern European thought, and the 'Afroasiatic roots of Classical Civilisation' have been neglected, discarded and denied.

Against this backdrop it is not surprising that that mathematics has been seen as the product of European mathematicians. However, there is now a widespread literature supporting the thesis that mathematics has been misrepresented in a eurocentric way, including Almeida and Joseph (2004), Joseph (2000), Powell and Frankenstein (1997) and Pearce (undated). A common feature of eurocentric histories of mathematics is to claim that it was primarily the invention of the ancient Greeks. Their period ended almost 2000 years ago, which was followed by the 'dark ages' of around 1000 years until the European renaissance triggered by the rediscovery of Greek learning led to modern scientific and mathematical work in Europe (and its cultural dependencies). This trajectory is illustrated in Figure 1.

Figure 1: Eurocentric chronology of mathematics history (from Pearce, based on Joseph, 2000, p. 4 Figure 1.1).

Some accounts have acknowledged the impact of lower level Egyptian and Babylonian mathematics on ancient Greek developments, as well as the later minor contributions of Indian and Arabic mathematicians (often seen primarily as custodians of Greek knowledge) on the history of mathematics in Europe (i.e., *The* history of mathematics). This is shown in the Modified Eurocentric model Figure 2).

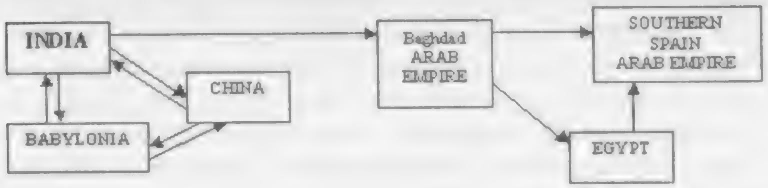
Figure 2: Modified Eurocentric model (from Pearce, based on Joseph, 2000, p. 9. Figure 1.2)



Pearce, Joseph and others go on to argue that in the so-called 'dark ages' and beyond, from 5^{th} - 15^{th} centuries, a great deal of mathematical work continued. Further the relationships between different regions and countries was complex and multidirectional and "A variety of mathematical activity and exchange between a number of cultural areas went on while Europe was in a deep slumber." (Joseph, 2000: 9). In Figure 3, I reproduce Pearce's diagram of interrelationships in the development of Non-European mathematics during the Dark ages.

Thus out of ignorance or prejudice arising from ideologically based values and preconceptions, eurocentric histories of mathematics, neglect the 'Non-European roots of mathematics' (to quote the subtitle of Joseph, 2000). There is a small but growing impact of such critical ideas in the history and philosophy of mathematics as indicated here. However, in my view, there is still an under-emphasis on the vital role of pre-Hellenic civilizations in providing the conceptual basis for modern mathematics through calculation, problem solving, etc.

Figure 3: Non-European mathematics during the dark ages (from Pearce, based on Joseph, 2000, p.10, Figure 1.3)



MATHEMATICS OF THE INDIAN SUBCONTINENT AND KERALA

One of the major casualties in the Eurocentric view of mathematics has been the ignoring or undervaluing of the contributions to mathematics of the Indian subcontinent. The long presence of deductive proofs in mathematics from this region has already been noted (Joseph 1994, 2000). Although the invention of zero by mathematicians of the Indian subcontinent has long been acknowledged, the significance of this as the lynchpin of the decimal place value system is often underestimated. Rotman (1987) presents a view of this innovation that puts its significance as reaching far beyond mathematics, right at the heart of European cultural and intellectual development in the Renaissance and early modern times. Pearce (undated) argues the Indian development of decimal numeration together with the place value system is the most remarkable development in the history of mathematics, as well as being one of the foremost intellectual productions in the overall history of humankind. I have indicated above how both philosophically and in the published histories of mathematics, calculation and numeration have traditionally been downplayed as epiphenomena of what is perceived to be the much more important Platonic conception of number. This is a misrepresentation of the intellectual significance of these developments without which the modern conceptions of number (including its computerization, with all of the applications this brings) would not be possible.

In the history of mathematics in the Indian subcontinent, much attention has been given to very large numbers, including powers of ten up to near 50. Whether these were contributors to or results of the development of the decimal place value system is for historians to say. Likewise it is tempting to speculate as to whether the extension of the decimal place value system into decimal fractions helped in the conceptualisation and formulation of the remarkable series expansions developed in Kerala. Although there is no unequivocal historical basis for this, it is convincingly claimed that floating point numbers were used by Kerala mathematicians to investigate the convergence of series (Almeida et al. 2001).

This brings me to one of the most remarkable and most neglected episodes in the history of mathematics, and the focus of this conference. This is the fact that Keralese mathematicians discovered and elaborated a large number of infinite series expansions and contributed much of the basis for the calculus, which is traditionally attributed to 17^{th} and 18^{th} century European mathematicians. Furthermore, this is not a case of simultaneous discovery in Kerala, for the work in Kerala took place two centuries before that in Europe.

Pearce (undated), Joseph (2000) and others attribute to Madhava of Sangamagramma (c. 1340 - 1425), the Keralese mathematician-astronomer, the important step of moving on from the finite procedures of ancient mathematics to treat their limit, the passage to infinity, the essence of modern classical analysis. He is also thought to have discovered numerous infinite series expansions of trigonometric and root terms, as well as for π, for which he calculated the value up to 13 decimal places (Pearce, undated). These inestimably important results anticipate some of the discoveries attributed to or named after the great mathematicians Gregory, Maclaurin, Taylor, Wallis, Newton, Leibniz and Euler.

Joseph (2000: 293) claims that "We may consider Madhava to have been the founder of mathematical analysis. Some of his discoveries in this field show him to have possessed extraordinary intuition". Almeida *et al.* (2001) have argued that Keralese contributions as a whole anticipate developments in Western Europe by several centuries in work on infinite series

for numerical integration results. In addition, these results are very possibly not just the anticipations of unacknowledged genius in the Indian subcontinent, and as such a very remarkable case of independent discovery. There is the very real possibility that these Keralese discoveries were transmitted to Europe by Jesuit missionaries and 'appropriated' by European mathematicians as their own (Almeida and Joseph 2004). The arguments for this transmission and appropriation are very persuasive, if not yet established with certainty. Certainly the mathematicians of Renaissance Europe are known to have been secretive about their methods and knowledge, and if they had 'purloined' the foundational results of calculus from Kerala would conceal and deny their origins.

As a non-historian of mathematics, I find this new recognition of the major Keralese and Indian subcontinent contributions to the history of mathematics remarkable. The fact that traditional histories of mathematics fail to acknowledge these and other non-European contributions is partly due to ignorance, for until recently it was difficult to find proper sources on this in standard texts. But there is much more to this, as there have been some reports of the anticipations in the literature for almost two centuries which have been disparaged or ignored. Instead there are two sets of entwined ideological presuppositions that have led to this denial and blindness. The first is the epistemological prejudice towards a certain style of mathematics, namely the axiomatic theories and purist ideas discussed above as well as favoring proofs over calculation al and applied mathematics. Through the lenses of these modern prejudices the historical contributions of non-Eurocentric traditions has been minimized and trivialized. The second set of ideological presuppositions is more sinister. This is the racial prejudice of Eurocentrism. Namely, that only the 'Western mind' (i.e., the Caucasian or European) is capable of the pure thought and insights required in the highest forms of mathematics. Thus the contributions of African, Asian, Indian subcontinent, and Oriental peoples are discounted and minimized, because by the presupposed 'very inferior nature' of these peoples they are incapable of the high levels of thought involved. Hence any results that contradict these prejudices is *ab initio* incorrect. Thus

such discoveries are minimized as intellectually inferior, or doubted and attributed to the transmission and copying of ideas from West to East, or in the last resort, challenged with regard to their chronology.

CONCLUSION

So what is the philosophical significance of the Keralese and Indian subcontinent contribution to history of mathematics? Identifying the most accurate genesis and trajectory of mathematical ideas in history that current knowledge allows should be the goal of every history of mathematics, and is consistent with any philosophy of mathematics. However, I have argued that a broader conceptualization of philosophy of mathematics is needed than the traditional emphasis on scholastic enquiries into epistemology and ontology. For such an emphasis has been associated, though I add need not necessarily be so, with an ideological position that devalues non-European contributions to history of mathematics. The philosophy of mathematics needs to be broad enough to recognise the salient features of the discipline it reflects upon, namely mathematics. As Lakatos (1976) indicated in the quote given above, the philosophy of mathematics has become empty by ignoring the history of mathematics.

It is no little charge to claim that the history and philosophy of mathematics have in effect become infused with error and a racist ideology, through the implicit and unacknowledged values and prejudices. Elsewhere, as well as above, I have argued that it is the business of mathematics and the philosophy of mathematics to take the issue of values seriously (Ernest 1998), and it is no longer enough to claim that these are outside of its proper subject matter. After all, ethics is just another branch of philosophy and I can see no grounds for its *a priori* exclusion. All human activities, however rarefied and abstruse are part of the vast cultural project of humankind, and as such none has the right to claim exemption from awareness of values and social responsibility (provided that this is not used as an excuse to limit freedom of thought and critique).

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DISCUSSION

This provocative paper aroused considerable interest and debate. The use of the terms 'racism' and 'eurocentrism' brought into the forefront of the discussion the issue of the relationship between knowledge and power: the mechanics by which colonialism 'overlooked to oversee, glorified to govern and denigrated to dominate'. The question at issue was whether the racism or eurocentrism that resulted was one of design or of ignorance. To answer this question in the Indian case required a closer study of knowledge as a 'necessary furniture of empire' which could in turn benefit from some of the insights offered by the sociology of knowledge. More specifically in relation to Kerala mathematics, there was a difference in opinion as to whether the Europeans had unintentionally suppressed the history or did so by design The colonialism in the European mind-set may have played a part in setting aside the Kerala contribution towards the development of modern mathematics. Maybe, a broader conceptualization of the philosophy of mathematics is required to pursue this debate any further.

There were more specific issues raised in the discussion. The paper imputes the refusal to acknowledge the influence of Kerala mathematics on European mathematics to epistemological and ethnocentric prejudices. Such an explanation can be credible only if it is first established that there was indeed such influence in the first place. However, there is the evidence of the earlier reticence in Europe to accept the use of the Indo-Arabic numbers to support such a conjecture; further later Arab developments in algebra and infinite series were largely ignored despite the European contiguity of Arab scholarship in the Iberian peninsula.[1]

Further, the paper also treats mathematical knowledge as fallible and non-foundational. Such arguments have been used to make modern scientific knowledge only Western local knowledge of the natural world and cannot therefore be deemed superior to the knowledge systems of other cultures that give a different picture of the universe. Fallible non-foundational epistemologies treat the knowledge of different cultures as incommensurable so that while they warn us against the

Eurocentric prejudice of making Western knowledge claims hegemonic, they also preclude combining together Western and non-Western traditions of knowledge. These constitute not complementary but alternative descriptions of the world.

The paper also implicitly assumes that Indian mathematical ideas and Western mathematical ideas can be combined – otherwise the former cannot have enriched and added to the latter. Hence, the fallibilist non-foundational position adopted opens the door to according equal respect to Western and Indian mathematics, but also closes the door to making them complement each other as part of a single system of global mathematics. The paper does not seem to address this implicit tension. Moreover, the epistemological similarities in the axiomatic and computational approaches suggested in the paper are intriguing and illuminating, but also suggest that we cannot treat Indian and European mathematics as simply alternative cultural constructions in the sense fallibilist non-foundationalism would suggest.

Finally, it was argued that the paper's conception of the parallels between the axiomatic and computational approaches is only credible for cases involving what can be described as 'finite mathematics'. In the case of infinite series expansions when we begin with a compound term, and obtain subsequent terms by application of computation rules, to finally reach the final term which is the numerical 'answer' to the problem, it cannot be said that the final term is numerically equal to the initial. It is only an approximation to the original term but not identical to it. By contrast in the case of finite mathematics all the terms in a sequence of derivations are strictly equal to each other so that the final numerical value is only a different way of expressing the value implicit in the original term. This suggests that the model of deductivism suggested by the paper is inadequate to represent, in its present formulation, the nature of computation after the passage to infinity.

These two levels of prejudice, these two value-based distinctions and preferences are frequently elided in the history and philosophy of mathematics. Thus the contributions of the ancient Greeks of the Euclidean type, and the modern focus on axiomatics of the past two centuries are seen to characterize the superior forms of thought of what is purported to be a Greco-European tradition. Furthermore, the unsystematized and unaxiomatized proofs and methods characterizing the official European history of mathematics from the late-Renaissance to the beginning on the Nineteenth century are seen as also reflecting the superior methods and concepts and higher forms of thought of the modern European tradition in their nascent phase, whose superiority and value is demonstrated in the subsequent flowering of the axiomatic tradition in Europe.

Through this elision, there arises the discounting of the proof-based contribution of cultures and civilizations outside of the 'Greco-European' tradition. Thus although there is a tradition of convincing demonstration or proofs, known as *Upapatis*, originating around two millennia ago in India, these proofs are discounted as intellectually inferior (Joseph 1994). Admittedly there are significant differences between the ancient Greek and the early Indian concepts of proof. Joseph (2000) has convincingly argued that ancient Indian mathematics was at least partially shaped by linguistic and grammatical conceptions of knowledge, based on the contributions of Panini; whereas ancient Greek mathematics was shaped by developments in philosophical thinking. So there are differences in the epistemological basis for different forms of proof that have emerged in different cultures and civilizations. However, the current challenges to the philosophy of mathematics discussed in the beginning of this paper, legitimate challenges to the traditional univocal and absolute conceptions of mathematics, knowledge and proof. From the perspective of the new fallibilist or social constructivist philosophies of mathematics, there is no ultimate or uniquely correct form of proof. Rather the forms of proof accepted within any culture or civilization during any epoch are a function of the historically contingent conceptual history and epistemological preconceptions that emerge and are accepted by the relevant geographico-historical communities of scholars. So there is no basis for elevating certain cultural forms of proof and demoting others on epistemological grounds alone. Each must be judged within the cultural contexts of its geographico-historical location.

ENDNOTES

[1] I take the beginning of disciplinary mathematics to be around 2500–3000 BCE, following Høyrup (1980) and (1994).

[2] Clearly branching derivations can occur in virtually all mathematical processes or activities including those involved in problem solving, deriving mathematical proofs, and mathematical calculations. However, they are mostly eliminated from the transcriptions of successfully completed activities.

[3] Note that I have not distinguished between the two analogous forms of proof which employ equivalence or deductive consequence as the transformational relationship at each step. In the latter the truth value derived is greater than or equal to is precedent value, in the former it is equal to it. But in each case (in bivalent logic), since the initial truth value in the sequence must be 1 the whole sequence of truth values including the final term, the theorem proved, is 1.

[4] The preservation of one of the four inequality relations along the sequence is possible variation, where an upper or lower bound on the value of the term is determined

[5] Technically the truth value function f can simply be defined on the domain of sentences under a given interpretation provided that there is an effective procedure for determining whether each sentence is true or false (thus giving values 1 or 0, respectively) under the given interpretation of the underlying theory or formal language.

[6] Joseph (2000) is among the few to note the importance of algorithms and calculation in the history of mathematics and to note their almost universal devaluation by other commentators.

PART – IV

Earlier Transmission of Mathematics from India and Parallel Developments Elsewhere

9

EARLY TRANSMISSIONS OF INDIAN MATHEMATICS

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1.0 Introduction

Scientific activity takes place in all cultures, sometimes independently of one another, other times from inspiration received from other cultures. The Renaissance notion of Greece being the sole origin of all sciences began to be shaken already towards the end of the nineteenth century, thanks to the efforts of early Orientalists. In recent times, the very lucid and therefore widely accessible *The Crest of the Peacock* contributed greatly in emphasizing the non-European roots of modern mathematics. It is understandable that as a reaction to the prevailing notion of Graeco-centrism, or Euro-centrism, other cultures too make similar claims; thus we have now Sino-centrism, Islam-centrism and Indo-centrism. One must naturally be cautious about these new movements; especially the last mentioned variety in its endeavour to read every scientific concept into the *Ṛgveda*.

But then how do we establish the origin of a scientific notion? One way is by the first occurrence of the notion in a particular culture, assuming of course that the chronology is secure. It may at least show that culture A, at an early period, was producing valuable scientific ideas. But priority is one thing; being the sole origin is another thing. The same idea may

have occurred to other people, say to culture B, quite independently, even if chronologically later. Therefore, to show that A transmitted the idea to B, or that B borrowed the idea from A, one must be able to trace the path of transmission from A to B and identify the facilitators of this transmission.

Sometimes, the receiving culture acknowledges the debt. For example, the Arabs have handsomely acknowledged that the decimal place value system and the associated symbols and operations came from India.[1] In his recent work *Wissenschaft und Technik im Islam,* Fuat Sezgin emphasizes that the Arab scholars of the medieval period scrupulously mention their sources.[2]

Indians were not far behind in their expressions of scholarly gratitude. Early Sanskrit texts on astronomy refer to the *Yavanas* (meaning Greeks) as masters in that subject. In an oft-quoted verse, Varāhamihira states that the Greeks, being great experts in astral science, are honoured like Ṛṣis.[3] Similarly, titles like *Romakasiddhānta, Yavanajātaka* suggest that these texts were derived from outside. In 1370 when the Jaina monk Mahendra Sūri wrote the very first Sanskrit manual on the astrolabe, he clearly states that this science of the astrolabe was derived from *Yavanas* (this time meaning Muslims).[4] A whole class of Sanskrit astrological works produced in this and the subsequent centuries on the basis of Islamic astrology were clearly designated as *Tājika.*[5]

Sometimes philology can come to the aid. For example, when we trace the derivation of the word *zero*, through medieval Latin *zephirum*, Arabic *ṣifr* to Sanskrit *śūnya,* or of the term "sine" through Latin *sinus*, through Arabic *jayb* to Sanskrit *jīvā*, this reflects the transmission of these terms and the concepts associated with them from India to Europe via the Middle East.[6] Loan words from foreign languages can indicate transmission of ideas; for example, the Greek words *hora, kendra* etc. in some Sanskrit texts.[7]

Sometimes the alien element clearly stands out. For example, oldest sources like the *Vedānga-jyotiṣa, Sūriyapaṇṇatti, Mahābhāṣya, Arthaśāstra, Śārdūlakarṇāvadāna* consistently state that the longest day has the duration of 18 *muhūrtas* (= 14;24 hours) as if this maximum value is true for all

geographical latitudes. Albrecht Weber suggested that this value, together with a host of other elements, was borrowed from Babylonia since Ptolemy gives almost the same duration (14 h 25 m) for the longest day at Babylon.[8]

But oftentimes, we can only talk of the "possibility" of transmission and not show conclusively that a transmission had actually occurred.[9] It is not my intention to give an overview of all the transmissions that took place before the advent of Kerala mathematics. These have been well documented, most notably by the author of *The Crest of the Peacock.* What I wish to discuss now are the problems one encounters when one tries to map the process and the paths of transmission. I shall do this through a few examples.

2.1 Development and Transmission of the Decimal System

I may begin with the most important of such transmissions, namely the transmission of the decimal place value system with the nine digits and zero, which is universally employed today. It is also universally acknowledged that it came from India. But problems start cropping up when one wishes to examine the different stages in the process of its development in India and in its of transmission from India.

There are four distinct elements that constitute the decimal place value system. Only when we are sure of the chronology of these elements within India, we can discuss their transmission outside. These elements are (1) counting with the base of ten, (2) the concept of zero, (3) the notion of place value, and (4) the symbols for the nine digits and zero.

Let us take the counting with the base of ten first. In the *Vājasaneyī Saṃhitā* of the Black *Yajurveda* and allied texts, there occur frequently series of decuple terms starting from *eka, daśa, śata* reaching up to *parārdha* which designates the thirteenth—and at that time, the ultimate— decimal place. Jaina canonical writings and Buddhist narratives contain much larger series. This terminology is quite fundamental to India; such large series did not exist anywhere outside India.[10]

Next, about the concept of zero (*śūnya*), Bibhutibhushan Datta and Avadhesh Narayan Singh have shown, as early as

1935, in their *History of Hindu Mathematics* that *śūnya* occurs for the first time in Piṅgala's *Chandaḥsūtra.*[11] Adequate attention has not been paid to this evidence in the prevailing historiography. I may, therefore, dwell on this aspect a little.

While teaching how to calculate the number permutations of a verse foot containing a certain number of syllables—each syllable being either short or long—, Piṅgala lays down a procedure in which certain steps in the calculation are to be marked with *dviḥ* (two) and certain others with *śūnya* (zero). That is to say, Piṅgala uses the symbols for zero and two as markers for distinguishing between two kinds of operations. The symbol of two marks the place where there is an even number which can divided by 2 and where squaring has to be done later; the symbol for zero marks the stages where there is an odd number and consequently absence of halving, and where a multiplication by 2 has to be performed. The whole computation can of course be done without any markers at all or with any other symbols. However, the fact that Piṅgala uses these two markers, that too in a meaningful way, shows that at Piṅgala's time there existed a well recognised symbol for *śūnya*, but we do not know what that symbol was. A symbol presupposes a concept. What kind of mathematical concept lay behind this symbol for *śūnya*? From Piṅgala's use, it may appear that *śūnya* meant here the absence of an operation, akin to the grammarian's *lopa*. But is that all, or does *śūnya* here imply place value as well?

In this connection, it is useful to consider the view of Joseph Needham who states that "Place value could and did exist without any symbol for zero. But zero symbol as part of the numerical system never existed and could not have come into being without place value."[12] Therefore, Piṅgala's employment of zero symbol pre-supposes place value. That it can only be decimal place value needs no emphasis. Therefore, the invention of the decimal place value system along with the concept and symbol for zero must antedate considerably Piṅgala's mention of the zero symbol. But when did Piṅgala live?

Albrecht Weber in 1863 and, following him, some scholars recently,[13] argued that the eighth chapter of the *Chandaḥsūtra* where *śūnya* occurs is not genuine because certain parts of this chapter do not occur in some manuscripts to which Weber had access. I have countered this by showing that "the eighth chapter containing combinatorics and the first mention of *śūnya* is not a late interpolation but an integral part of Piṅgala's *Chandaḥsūtra.*"[14] It is not necessary to repeat my arguments here.

But Piṅgala's time is difficult to determine in absolute terms, and here lies one of the serious problems of Indian historiography. All that can be said is that there exists a close similarity in the method of exposition and in the mode of notation between the *Chandaḥsūtra* on the one hand and the *Aṣṭādhyāyī* and the *Vedānga-jyotiṣa* on the other. This affinity in style places the *Chandaḥsūtra* about 400 BC; in any case certainly before the commencement of the Christian era. Therefore it will not be rash to conclude that the decimal place value system with a symbol for zero developed in India before the beginning of the Christian era.

Babylonians had place value in their sexagesimal system, though not the zero. The Chinese too had a "fundamentally decimal place value" much before the third century AD, but they too did not have the zero. Nevertheless, could the Babylonian notion of place value and the Chinese decimal system have played any part in the development of the decimal place value in India? It is not impossible, though firm evidence is lacking. This much, however, seems to be certain: the full development of the decimal place value in its modern form with the use of zero took place in India before the beginning of the Christian era— not in the fifth century as most of the current literature states— and it is from India that the system was transmitted to other countries.

Bibhutibhusan Datta and Avadhesh Narayan Singh in their above-mentioned book, and several subsequent writers enumerated the literary evidences for decimal place value in the early centuries of the Christian era, such as *Anuyogadvāra* (probably 100 BC), Sphujidhvaja's *Yavanajātaka* (269/70 AD), *Lokavibhāga* (458 AD) and so on.

But what about the written symbols for the nine digits and especially for zero? It is one of the ironies of history of Indian science that the earliest written records containing a symbol for zero are found not in India but outside. In a valuable paper, G. Coedès discussed the numerical expressions and symbols

occurring in the inscriptions of South-East Asia between the fifth and eighth centuries AD.[15] In one of these inscriptions, viz. in a Sanskrit inscription found at Bayan in Cambodia, the Śaka year 526 (= 604 AD) is recorded in numerals as "526" and in word symbols as "*rasa-dasra-śaraiḥ śakendravarṣe*". The decimal place value is clearly evident both in the numerals and in the chronogram with the word symbols. Therefore, it is argued that this is the earliest occurrence of numerals with place value and that, therefore, the modern system of decimal place value originated in South-East Asia from where it spread to India.

But Datta and Singh drew attention to an earlier occurrence in India itself. This is the Mankhani charter found at Sankheda in Gujarat. Here the year is written in numerals in decimal notation as "346"; the year is in the *Kalachuri-Chedi* Era, corresponding to 594-6 AD.[16] Thus there is conclusive evidence for the use of numerical symbols with place value in India, even before the inscription in Cambodia.

Still doubts persist about the origin of the zero symbol and its use with decimal place value. In the paper mentioned above, Coedès refers to two inscriptions that contain zero symbols. A Khmer inscription at Sambor in Cambodia from AD 683 records the Śaka era 605, representing the zero with a thick dot (•); three years later, a Malay inscription at Palembang in Sumatra from AD 686 contains the numbers 60 and 608 Śaka, where zero is shown by a small circle as we write today (o).

Does this warrant the conclusion that the symbol for zero developed in South-East Asia and then spread to India, as Needham argues? It is well worth quoting Needham in full:

> "Coedès does not believe that the south-east Asian inscriptions indicate an east Asian origin for the symbolic word system..., but rather that the Hinduising settlers of south-east Asia already had symbolic words and the old numerals when they first went there, or at any rate were soon followed by them. So far so good, but we are free to consider the possibility (or even probability) that the written zero symbol, and the more reliable calculation which it permitted, really originated in the eastern zone of Hindu culture where it met the southern zone of the culture of the Chinese. What ideographic stimulus could it have received at that interface? Could it have adopted an encircled vacancy from the empty blanks left for zero on the Chinese counting boards ? The essential point is that the Chinese had possessed, long before the time of *Sun Tzu Suan Ching* (late +3rd Century) a fundamentally decimal place-value system. It may be, then, that the 'emptiness' of Taoist mysticism, no less than the 'void' of Indian philosophy, contributed to the invention of a symbol for *śūnya*, i.e. the zero. It would seem, indeed, that the finding of the first appearance of the zero in dated inscriptions on the border line of the Indian Chinese culture-areas can hardly be a coincidence."[17]

Unfortunately, there is more wishful thinking than reasoned argument in this passage. In his eagerness to prove the Chinese inspiration for the invention of zero, Needham completely ignores the occurrence of *śūnya* in Piṅgala's *Chandaḥsūtra*. He ignores too the other Sanskrit inscriptions discussed by Coedès in the same article. But, if the symbol for zero was invented in South-East Asia in the seventh century under inspiration from China—which did not until then had a symbol of her own— and then transmitted to India, why then were two symbols invented at the same time in South-East Asia, a small circle and a thick dot? It should also be noted that there are several other inscriptions in this region which were composed in Sanskrit; which show that the Indian settlers brought with them Sanskrit language, Indian calendar, Indian Śaka era, decimal place value, Sanskrit word numerals, Indian symbols for the digits 1 to 9. Surely they would also have brought the zero symbol with them.

The fact is that these two symbols developed in India, not at the same time, but one after the other. Of these, the dot (*bindu*) is the earlier one. Later it was enlarged into a circle (*chidra* or *randhra*) for purely practical reasons; so that it could be clearly recognised as a symbol and not mistaken for an accidental dot. An analogous shift occurred with *anusvāra* in Kannada-Telugu script. In early inscriptions, the *anusvāra* was represented by a small dot, often hardly visible on the copper plates. Gradually it was replaced by a small circle, which is called in Telugu *sunna*

(from Sanskrit *śūnya*) since it was identical with the symbol for *śūnya*.

Since these two symbols appear in inscriptions of South-East Asia in the seventh century, it is reasonable to conclude that both the symbols were in use in India at this time and that the circular symbol must have developed by this century at the latest. The change, however, was not simultaneous everywhere. The dot seems to have prevailed longer in Gandhāra and Kashmir regions. Thus in the mathematical text known as the *Bakhshālī Manuscript*[18] and in the unique copy of an anonymous commentary on Śrīdhara's *Pāṭīgaṇita,*[19] both written in an early form of Śāradā script some time after the ninth century, the zero is represented by a dot (.). Moreover, even after the dot was replaced by the circle, the symbol continued to be called *bindu* or *śūnya-bindu.*[20]

While the symbol for zero was gradually transforming itself from a dot to a circle, the symbols for the nine digits too underwent changes. These changes can be seen in epigraphic records in India as well as in South-East Asia.

2.2 Transmission to China

With the spread of Buddhism in China, there took place a massive exchange of scholars between India and China. Buddhist pilgrims visited India and carried back large quantities of manuscripts with them; Indian scholars went to China where they were active in translating Buddhist texts into Chinese. These Chinese and Indian scholars were primarily interested in religious texts, especially of Buddhism, and the reports generally talk only about religious and philosophical texts. But they must have also carried with them some notions or texts about Indian systems of numeration, mathematics and astronomy. About this activity there are some stray references.

The catalogue of the Sui dynasty, completed in 610 AD, mentions some Chinese translations of Indian works on astronomy, mathematics, and medicine. These works are now lost, but their very existence shows that towards the end of the sixth century, the Chinese had gained some knowledge of Indian astronomy, mathematics and pharmaceutics.

Records of the Tang dynasty indicate that from 600 AD onwards Indian astronomers were employed at the Astronomical Board of Chang-Nan to teach the principles of Indian astronomy and calendar. One of the Indians named Gautama Siddhārtha was reported to have constructed a calendar, based on the Indian *Siddhāntas*. This calendar contains sections on Indian numerals and arithmetical operations and sine tables at intervals of 3°54' for a radius of 3438 units, which are precisely the values given in the Indian astronomical texts. There survives a block print text, which contains Indian numerals, including the use of a dot to indicate zero.[21]

Strangely enough, neither the dot for zero found in this calendar prepared by Gautama Siddhārtha, nor the dot and circle for zero found in inscriptions in what Needham refers to as the "southern zone of the culture of the Chinese" in the seventh century seem to have had any impact in China proper, because the Chinese did not start using the zero symbol until the mid-thirteenth century when it appears for the first time in the work of Qin Jiushao. It is rather intriguing why the Chinese took such a long time (nearly six centuries) to use the zero symbol which was brought up to their doorstep, so to speak. Here is a case of clear transmission having no impact in the receiving culture. The same appears to be the case with the other elements of Indian mathematics and astronomy introduced into China in the Tang period.[22]

To conclude, there are extensive records about the transmission of Buddhist texts and ideas from India to China. There are also records of Sanskrit astronomical and mathematical texts being translated into Chinese. In the reverse direction, we have no records of transmission from China to India. Still transmission of ideas must have occurred as did the transmission of objects. A number of parallel developments are also known. But one looks in vain for firm evidence of interaction between these two cultures in the realm of mathematics. This then is a grey area in the history of intellectual exchanges between these two cultures.

2.3 Transmission to the West

Fortunately, such uncertainties do not occur in the case of transmission to the Middle East or the Arab culture area. During

the reign of the second Abbāsid Caliph al-Mansūr (753-774), the Indian province of Sindh passed under the control of the Caliphate and embassies from Sindh started visiting Baghdad. Sometimes these were accompanied by scholars. In his *India,* Al-Bīrūnī states : "These star-cycles as known through *the canon* of Alfazârî and Ya'ḳûb Ibn Târiḳ, were derived from a Hindu who came to Baghdad as a member of the political mission which Sindh sent to the Khalif Almansûr, A.H. 154 (= AD 771)."[23] Others also report about this mission. Ibn al-Adamī states in the preface to his astronomical tables entitled *Naẓm al-iqd* that "an Indian astronomer, well versed in the sciences, visited the court of al-Mansūr, bringing with him tables of the equations of planets according to the mean motions, with observations relative to both solar and lunar eclipses and the ascension of signs. Abu-Māsher of Balkh, an astrologer at the court of al-Mansūr, mentions that he derived the knowledge of the Hindu great cycle of the '*kalpa*' from an Indian astronomer. The name of this Indian astronomer is written variously as 'Kankaraf', 'Kankah', or 'Cancah', 'Kenker'...."[24] David Pingree avers that the name was Kanaka and that "the later Arabic writers slowly developed an elaborate mythology concerning Kanaka's role in the history of astronomy," attributing to this mythical figure scholarship and skills of diverse kinds.[25]

Perhaps the real word was not "Kanaka" but "Gaṇaka"— not a proper name but a generic name for the astronomer. Probably this word referred not to one particular astronomer who visited Baghdad in 771, but collectively to all Hindu astronomers or learned people who visited Baghdad about this time. This would explain the diverse qualities attributed to this Gaṇaka who may have, in reality, represented different persons.[26]

Be that as it may, the Indian decimal place value system reached the Middle East not through this visit of Kanaka / Gaṇaka but at least a century earlier, for already in 662 AD the Nestorian Bishop Severus Sebokht sings the praise of Indian decimal numbers, in a passage that is oft-quoted.[27]

The embassies from Sindh to the court of Al-Mansūr resulted in the transmission of many more scientific ideas, besides the decimal system and arithmetical operations with this

system. At the court of Al-Mansūr, Brahmagupta's *Brāhmasphuṭasiddhānta* and *Khaṇḍakhādyaka* were rendered into Arabic respectively by Al-Fazārī and Ya'qūb ibn Tārikh. These are not literal translations but adaptations and came to be known under the names *Sindhind* and *Al-Arkand.* Through these works, Arab scholarship became acquainted for the first time with mathematical astronomy, a few decades before the discovery of Greek astronomy. In the next century, around 820 AD, Muḥammad ibn Mūsa al-Khwārizmī summarized the knowledge gained thus far in three works, one on arithmetic, another on algebra and the third on mathematical astronomy. In the first work on arithmetic entitled *Kitāb al-jam'wal tafrīq bi ḥisāb al-Hind,* al-Khwarizmī explained the arithmetical operations of addition, subtraction, multiplication, division and the extraction of square roots according to the Indian system. Within a century, this treatise was superseded in the Eastern Islamic world by other introductions to Indian arithmetic. However, the work was still available in the Moorish Spain in the twelfth century where it was translated into Latin, under the title *Liber algorismi de numero Indorum,* "The Book of al-Khwārizmī on Indian numbers." Soon European scholars recognized the value of al-Khwārizmī's treatise and there appeared more treatises in Latin elaborating al-Khwārizmī's treatise. What needs to be emphasized in this context is that the symbol for zero known to al-Khwārizmī and was transmitted through him to Europe was a small circle.[28] Indeed zero is called in Arabic, besides *al-ṣifr,* also by the expression *dā'ira ṣaghīra* (small circle). And it is this small circle (*circulus*) which appears in early Latin manuscripts.[29] While zero retained its circular form in its transmission from the Eastern Islamic world to the Western Islamic world, the digits from 1 to 9 gradually underwent several changes, with the consequence that the western forms substantially differed from the eastern ones. These western forms came to be known as *Ghubār* numerals. The eastern forms were transmitted to Italy and the Eastern Mediterranean basin where they were used by Latin authors in the twelfth century. By the early thirteenth century these eastern forms were replaced in Europe by the western forms through the Latin translations made in the Moorish Spain.[30] These then are

the ancestors of what we call today "Arabic" numerals (or what the Constitution of India, more accurately and wisely terms, in article 343.1, as "the international form of Indian numerals"[31]).

As mentioned earlier, the zero retained its form in Europe in the shift from eastern to the western numerals. But in the Arabic script, apparently at a later point, this circle was compressed into a dot, in order to distinguish the symbol for zero from that of five —a process that is the reverse of what happened in India.

But the westward transmission of Indian mathematical ideas was not limited to the number system alone. It has been stated that the main areas which influenced the future course of development of mathematics are (1) the spread of Indian numerals and their associated algorithms, first to the Arabs and later to Europe; (2) the spread of Indian trigonometry, especially the use of the sine function, and (3) the solutions of equations in general, and of indeterminate equations in particular. In this context, I may invite attention to a special case of transmission of a rather simple procedure of calculation, namely *Trairāśika* or the Rule of Three.

3. Development and Transmission of the Rule of Three

Though normally employed in solving commercial problems, such as computing the price of *a* mangoes, if the price of *b* mangoes is known as *c* Rupees, *Trairāśika* or the Rule of Three plays a far more important role in Indian mathematics and astronomy.[32] In arithmetic it is often used as a means of verification in solving other problems. More importantly, it is employed in astronomical computations, for example, in the computation of the mean position of a planet from the number of its revolutions in a *kalpa* of 4,320,000,000 years. Many of the problems of spherical trigonometry are solved by applying the Rule of Three to similar triangles (*akṣakṣetra*). Likewise, the Rule of Three forms the basis for computing trigonometric ratios. Therefore Nīlakaṇṭha Somasutvan declares in his commentary on the *Āryabhaṭīya* that the entire mathematical astronomy (*graha-gaṇita*) is pervaded by two fundamental laws: by the law of relation between the base, perpendicular and hypotenuse in a right-angled triangle—which goes today under

the name of Pythagoras theorem— and by the Rule of Three (*bhujakoṭikarṇanyāyena trairāśikanyāyena cobhābhyāṃ sakalaṃ grahagaṇitaṃ vyāptam*).

Therefore it would be interesting to trace its development and spread. In India the Rule was first mentioned by Āryabhaṭa in his *Āryabhaṭīya* in 499 AD.[33] Here Āryabhaṭa not only gives the name *Trairāśika* (that which consists of three numerical quantities or terms) for the Rule of Three, he also mentions the technical terms for the four numerical quantities involved (*pramāṇa, phala-rāśi, icchā-rāśi, icchā-phala*) and gives the formula for solving the problem. Subsequent writers, notably Brahmagupta in his *Brāhmasphuṭasiddhānta* (628 AD) and Bhāskara I in his commentary (629 AD) on the *Āryabhaṭīya* elaborate upon this brief statement by Āryabhaṭa, but employ the same terminology, albeit with slight modifications. It is on the basis of the writings of these mathematicians that histories of mathematics generally trace the origin of the Rule of Three to India.

The brief manner in which Āryabhaṭa presents the rule in his work implies that he is referring to an already well known rule which he is restating in order to employ it in astronomical computations. Therefore, it is tempting to look for the antecedents for Āryabhaṭa's rule.

Kuppanna Sastry sees the first mention of the Rule of Three in a verse of the *Vedāṅgajyotiṣa* (Rk-recension 24; Yajus-recension 42).[34] Obviously, we have here a rudimentary form of the Rule of Three and it is also obvious that the rule was needed for the computations envisaged in the text. Although Indians developed special terminology for the Rule of Three in later times, the general terms *jñāna<ta>rāśi* (the quantity that is known or given), *jñeya-rāśi* (the quantity that is to be known) used here are also frequently employed in later times. Indeed it is conceivable that the term *jñāna* gave rise to the later term *pramāṇa*. However, the date of this text, available in two recensions, is uncertain. Kuppanna Sastry himself would like to place the composition of the text in the period between 1370 – 1150 BC; others assign it to 500 BC. In either case, Āryabhaṭa's rule appears to have a long pre-history in India.

However, Joseph Needham observes that the "Rule of Three, though generally attributed to India, is found in the Han *Chiu Chang*, earlier than in any Sanskrit text. Noteworthy is the fact that the technical term for the numerator is the same in both languages --- *shih* and *phala*, both meaning 'fruit'. So also for the denominator, *fa* and *pramāṇa*, both representing standard unit measures of length." Needham goes on to add that "Even the third known term in the relationship can be identified in the two languages. For *icchā*, 'wish, or requisition' reflects *so chhiu lü*, i.e. ratio, the number sought for." [35]

Writing in the *Gaṇita-Bhāratī,* N. L. Maiti, draws attention to the passage of the *Vedāṅga Jyotiṣa*, in order to counter Needham's claim of Chinese priority.[36] Maiti also disputes Needham's linguistic equation *fa = pramāṇa; shih = phala; so chhiu = icchā.* Finally, he tries to clinch the issue by citing the view of a Chinese scholar Lam Lay Lang to the effect that "the Rule of Three ... originated among the Hindus..."[37]

I am not competent to judge in favour or against Needham's linguistic equation, but there is no denying the fact that the Rule of Three had an important place in Chinese mathematics as well. Even if the verse from the *Vedāṅgajyotiṣa* alludes in a rudimentary form to the Rule of Three and thus testifies to the existence of the rule in the centuries before the Christian era, there is nothing to prevent the knowledge of the Chinese *Chiu Chang* to travel to India in the early centuries of the Christian era and to give impetus to the development of the Rule in India. If India received impetus from China in this process of development and then transmitted an elaborate system to the Middle East and Europe, then this would testify to both the receptivity and creativity in mathematical thought in India. It would, however, be nice if the transmission could be mapped in detail.

By the time of Brahmagupta in the early seventh century, the Rule of Three and its variations reached their full development. In the next century, various elements of Indian mathematics and astronomy were disseminated to the Islamic world. The Rule of Three seems to be one of the elements thus transmitted. From the ninth century onwards, Arab mathematicians began to discuss the Rule of Three and other variants.

Thus Al-Khwārizmī discusses the Rule of Three in his book on Algebra. This treatise contains a small chapter on commercial problems including the simple Rule of Three according to the Indian model.[38] Al-Bīrūnī (973-1048) composed an exclusive tract on the Rule of Three entitled *Fī Rāshikat al-Hind,* where he discusses direct and inverse Rule of Three as well as the rules for five, seven and more terms up to seventeen.[39] Together with Indian numerals and commercial problems, the Rule of Three was transmitted to Europe where it was hailed as the Golden Rule.[40]

This case too exemplifies that while it is possible to trace the path of transmission between India and the West, the historiography of mathematics has not yet reached a stage where it could clearly define the interaction between Chinese and Indian mathematics and astronomy.

4.0 Transmission of Perpetuum mobile

I should like to discuss another case where linguistic limitations and national or personal biases obstruct the correct apprehension of the transmission of ideas, this time of technical nature.[41] In the context of non-European contribution to the development of technological ideas in Eurpe, Lynn White brought out significant studies on medieval technology. Notable in this connection is his seminal essay *Tibet, India, and Malaya as Sources of Western Medieval Technology.*[42] One of the concepts whose origin he attributes to India is the perpetual motion machine. Lynn White avers that the perpetual motion machines designed by Bhāskara in the twelfth century were instantly accepted by the Islamic world and then transmitted to Europe, where people like Villard de Honnecourt, in their quest for energy, received this notion with great interest and tried to apply it to the benefit of mankind. Thus, concludes White, were laid the foundations for the power technology of the modern world.

Lynn White's thesis was generally accepted by historians of technology, but his attempt to trace the origin of the perpetual motion machine to twelfth-century India was contested by two sides, the former contending that such machines were known to the Arabs before Bhāskara's time and the latter claiming that

both the Indian and Arabic accounts owe their inspiration to China.

Ahmad Y. Al-Hassan and Donald R. Hill argue in their excellent book *Islamic Technology: An Illustrated History* thus: "In India about A. D. 1150 Bhāskara described a perpetual motion wheel which resembles one of the six such wheels in the Arabic manuscripts, but the original Arabic text is of an earlier date. The Arabic technical descriptions, the illustrations, and the whole complex of the sixteen machines are quite elaborate and, as we have seen, constitute a single approach. The occurrence, therefore, of one or two perpetual-motion wheels in the Indian text does not imply a case of transmission from one culture to another, though there was an important transmission to the West."[43] About the original Arabic text, the authors maintain: "It must have been copied from an original treatise which is at present unknown to us. We can tell, however, that this original was written between the third and sixth centuries AH (ninth to twelfth centuries AD)."[44] But it is possible to show that the Indian concept of the *perpetuum mobile* is much older than Bhāskara and also older than the alleged antiquity of the "unknown and undated" original Arabic text.

The second party of opposition to Lynn White's view is represented by Joseph Needham, who asserts that "Indeed one begins to entertain the belief that the stimulus for the flood of ideas on the perpetual motion devices may have been derived from Indian monks or Arabic merchants standing before a clock tower such as that of Su Sung and marvelling at its regular action."[45] Lynn White dismissed this suggestion as lacking in any evidence.[46]

The astonishing thing about this debate—like many other debates concerning India's past—is that it is conducted on the basis of just two Sanskrit texts which happen to be available in English translation, ignoring all other texts. Lynn White traces the idea of the perpetuum mobile to twelfth-century India on the basis of Lancelot Wilkinson's translation of the *Siddhāntaśiromaṇi*,[47] while Needham's comments emanate from his perusal of Ebenezer Burgess's rendering of the *Sūryasiddhānta*.[48] The passage cited by Needham does not even discuss the perpetuum mobile. No doubt, Lynn White's conclusions are highly perceptive even with the limited sources available to him, but in history of

technology there are no shortcuts. One has to study all the relevant original texts, and the material remains if there are any, and then interpret the data in the correct space-time framework. In the present case, a study of the original texts not only upholds Lynn White's view, but strengthens it further. For a perusal of the Sanskrit sources shows that the idea of perpetual motion is much older in India than Bhāskara's time and that the philosophical notion was translated into a design for a mechanical instrument by Brahmagupta in the early seventh century. Brahmagupta's mercury wheel is not only earlier than Su Sung's clock tower (1090 AD) but also works on a totally different principle. On the other hand, there is much in common between Brahmagupta's wheel and those found in Arabic manuscripts, for both are supposed to be driven by mercury power. As mentioned earlier, Brahmagupta's writings on astronomy and mathematics were transmitted to the Islamic world in the eighth century and these may have included the design for the perpetual motion machine as well.

"In today's world of narrow loyalties, one is accustomed to ask to whom the credit should go: is it due to Brahmagupta for the origin of the idea, or to the Islamic world for its elaboration and spread, or to the Occident for its practical application? Lynn White, quite rightly, sees these three kinds of endeavour as complementary to one another."[49]

5.0 Transmissions within India

While the study of transmissions from and to India has its own importance, I think it is also necessary to study the transmission of ideas within India, from one region to another, from Sanskrit to regional languages and vice versa.

On the one hand, we have the cases of swift transmission of ideas from one end of the country to the other. In 1185 AD, in his commentary on the *Sūryasiddhānta*, Caṇḍīśvara of Mithila cites another commentary on the same text written by Mallikārjuna Sūri in the distant Telugu region just 7 years previously in 1178 AD.[50] On the other hand, certain texts like the *Candra-vākyas* of Vararuci do not seem to have been

transmitted beyond Tamilnadu.[51] Nearly all the surviving manuscripts of the *Āryabhaṭīya* are only in Malayalam script.

Again, certain texts, indeed very valuable ones, are irretrievably lost, although there had been no significant break in the study of mathematics and astronomy. It is greatly intriguing why Āryabhaṭa's book on the midnight reckoning system is wholly lost; why Bhāskara's excellent commentary on the *Āryabhaṭīya* is just partially preserved; and why the equally excellent commentary on Śrīdhara's *Pāṭīgaṇita* is not completely available. Indeed, Mādhava's contributions to the power series are known only as citations from later writers; what happened to Mādhava's original works ?

Even such a useful tool like the *Kaṭapayādi* notation, which greatly facilitated the development and spread of astronomical and mathematical tables in Kerala, did not spread or spread very slowly to other parts of India.[52] That it was employed in Kerala very widely, not only in works on astronomy and mathematics, but also in non-scientific works, not only in Sanskrit writings but in Malayalam as well, is now quite well established.[53] From Kerala, the *Kaṭapayādi* system spread to the neighbouring Tamilnadu, where its use is fairly well attested.[54] However, no literary works from the regions of Karnataka or Andhra have been identified so far which employ this system of notation.

Towards the middle of the tenth century Āryabhaṭa II used a variant[55] of the *Kaṭapayādi* system in the first thirteen chapters of his *Mahāsiddhānta.* Here the digits are read from the left to the right and every member of a conjunct consonant has a numerical value.[56] But we do not know where this Āryabhaṭa hailed from. Two centuries later, in Maharashtra, Bhāskara II employed this notation a few times in his commentary on the *Śisyadhīvṛddhida,*[57] but not in any other work of his. In North India proper, I know of only two instances. These occur in the works of Rāmacandra Vājapeyin[58] and his brother Harṣa,[59] who flourished in the first half of the fifteenth century in Naimiṣāraṇya, close to modern Sitapur in Uttar Pradesh. These are the only instances from literary documents. The *Bhūtasaṃkhyā*[60] system of word numerals is widely employed in inscriptions,[61] but the use of the *Kaṭapayādi* system is limited to very few records, that too belonging to the 14-16th centuries.[62]

While cataloguing pre-modern Indian astronomical instruments, I came across two Sanskrit instruments that carry the *Kaṭapayādi* notation. In Islamic astrolabes and celestial globes, it is customary to inscribe the numbers on the scales in an alphabetical notation called *Abjad.* Clearly in imitation of this, an undated Sanskrit astrolabe, now preserved in the Sanskrit University at Varanasi, displays *Kaṭapayādi* notation in some of the scales.[63] In the middle of the nineteenth century, a certain Bhālumal of Lahore, who made astrolabes and celestial globes in Arabic as well as in Sanskrit, produced a Sanskrit celestial globe on which the scale on the horizontal ring was labelled in the *Kaṭapayādi* notation. This globe is now in a private collection in Milan.

Likewise, transmissions from Sanskrit to regional languages and vice versa remain an almost unexplored area. Thanks to the bibliographical work of Professor K. V. Sarma, we know about mathematical and astronomical works in Malayalam language, but very little is known about such texts in other Indian languages. Without exploring the literature in regional languages, a full picture will not emerge of how mathematical ideas developed and systematized in Sanskrit manuals and how they were disseminated and popularized in the regional languages.[64]

DISCUSSION

This paper examines some cases of possible early transmissions of Indian mathematical ideas (the decimal place-value number system with zero, the rule of three and the notion of perpetual motion machines) to other cultures – in particular Chinese, Arabic and Western civilizations. It also tries to address some epistemological and methodological issues concerned with documenting and establishing such occurrences. Moreover, given the geographical extent and linguistic diversity of India, it points to the importance of studying how mathematical ideas were disseminated historically within India itself. The paper's general conclusion is that although ample documentation exists for Indian transmissions to the Arab world, and thence to the West, there is much less documentation or study of other

transmissions. It recommends that more attention be paid to them in the future.

The paper's key contribution appears to be the controversial claim that the decimal place-value system with zero developed in India before the beginning of the Christian era rather than in the fifth century as is often supposed. However, the argument made for this realignment of mathematical history does not stand close scrutiny. To see why such a conclusion is warranted let us examine case made by the paper more carefully.

The paper begins by identifying the distinct elements constituting the Indian number system - counting with base ten, the notions of zero and place value, and the symbols used to represent the nine digits and zero. All of these elements, the paper argues, can be traced to India before the Christian era. Firstly, decuple terms occur in Hindu and Jain religious canonical texts in that period proving that Indian mathematics adopted a decimal system.

Secondly, the term *śūnya* occurs in Pingala's *Chandahsutra* in the context of teaching a method for computing the number of permutations of a verse containing a certain number of syllables, where Pingala uses the symbol for *sunya* to denote the *absence* of the mathematical operation of halving. The paper goes on to ask "What kind of mathematical concept lay behind this symbol for *sunya*? From Pingala's use, it may appear that *śūnya* meant here the absence of an operation, akin to the grammarian's *lopa*. But is that all, or does *śūnya* here imply place value as well?"

However, instead of answering this question the paper appeals to Needham's view that the zero symbol could not have come into being without knowledge of place value. The inference is that Pingala's employment of the zero symbol presupposes knowledge of place value. Consequently, the decimal place value system with the concept of zero must have been known to Pingala.

However, it could be argued that the paper presupposes what it sets out to prove – namely that Pingala had the concept of zero. Even if Needham rightly assumes that the use of the symbol *sunya* for zero presupposes knowledge of place-value, Pingala's use of the symbol *sunya* for an absence of an operation does not need to presuppose the concept of place-value. Hence

Pingala's use of the symbol *sunya* to denote 'absence of an operation' cannot be used to infer that he had knowledge of the concept of zero.

ENDNOTES

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1 Thus Muḥammad ibn Mūsā al-Khwārizmī (ca. 780-850) calls his book *Kitāb al-jam'wal tafrīq bi ḥisāb al-Hind,* "Book on Addition and Subtraction after the Method of the Indians" and Abul l-Ḥasan al-Uqlīdīsī his book, written in 952, *Kitāb al-fuṣūl fi-l-ḥisāb al-Hindī,* "The Book of Chapters on Indian Arithmetic".

2 Fuat Sezgin, *Wissenschaft und Technik im Islam,* Frankfurt, 2003, Vol. 1, p. 142: "In der bisherigen Historiographie der Wissenschaften wurde leider zu wenig beachtet, dass das Zitieren von Quellen eine der charakeristischen Eigenschaften des arabisch-islamischen Schrifttums ist, auch wenn dies nicht bedeutet, dass es dort keine Plagiate gegeben hätte oder sich jeder Schriftsteller an die allgemeine Regel gehalten hätte."

3 *Bṛhatsaṃhitā,* (ed. Avadh Vihari Tripathi, Varanasi, 1968), 2.75:
mlecchā hi yavanās teṣu samyak śāstram idaṃ sthitam /
ṛṣivat te 'pe pūjyante kiṃ punar daivavid dvijaḥ //

4 *Yantrarāja,* (ed. Kṛṣṇaśaṅkara Keśavarāma Raikva, Bombay, 1936), 1.3:
kḷuptās tathā bahuvidhā yavanaiḥ svavāṇyā yantrāgamā nijanijapratibhāviśeṣāt /
tān vāridhīn iva viloḍya mayā sudhāvat tatsārabhūtam akhilaṃ praṇipadyate 'tra //

5 David Pingree, *Jyotiḥśāstra: Astral and Mathematical Literature,* Wiesbaden 1981, pp. 95-100.

6 While the transformation of Sanskrit *jīvā* into Arabic *jayb* is understandable, it is difficult to account for the change of *śūnya* to Arabic *ṣifr*.

7 Needless to say that mere phonetic similarity of certain terms does not establish the transmission of ideas unless it is supported by other evidence.

8 A. Weber, *Die Vedischen Nachrichten von den Naxatra (Monstationen),* part 2, Berlin, 1860-1862, pp. 362-63. The *Report of the Calendar Reform Committee,* New Delhi, 1955, pp. 225-26, and Kane, *History of Dharmaśāstra.* Vol. I, part 1. Poona, 1958, pp. 541-43, contest this view, with arguments already anticipated by Weber. Kane asserts that any astronomer living in Gandhāra could have arrived at the longest day of 18 *muhūrtas* after a few

years of observations without having to borrow the idea from Babylonia. While that is possible, there is no record of maximum and minimum values from other latitudes, say from that of Ujjain. It is striking that all these texts (and several *purāṇas* cited by Kane, p. 41) should consistently speak only of the longest day of 18 *muhūrtas*, without allowing for changes according to the local latitude, as though they were repeating a hoary tradition rather than their own observations.

9 Therefore, I particularly appreciate efforts of the Āryabhaṭa group in looking for evidence that is "legally binding" for the transmission of Kerala mathematics to Europe. Although the possibilities do exist, the path of transmission can be traced and possible mediators can be identified, still it is necessary to show that the transmission did indeed take place.

10 K. V. Sarma & B. V. Subbarayappa, *Indian Astronomy : A Source-Book,* Bombay, 1985, pp. 46-47; Takao Hayashi (ed. & tr), *The Bakhshālī Manuscript: An ancient Indian mathematical Treatise,* Groningen, 1995, p. 66.

11 Bibhutibhusan Datta & Avadhesh Narayan Singh, *History of Hindu Mathematics: A Source Book,* Bombay, 1935, 1939; reprint, 1962, part I, pp. 75-77; see also S. R. Sarma, "Śūnya, Mathematical Aspect" in: Bettina Bäumer (ed), *Kalātattvakośa: A Lexicon of Fundamental Concepts of the Indian Arts*, vol. 2, New Delhi, 1992, pp. 400-411; idem, "Śūnya in Piṅgala's Chandaḥsūtra" in: A. K. Bag & S. R. Sarma (ed), *The Concept of Śūnya,* New Delhi, 2003, pp. 126-136.

12 Joseph Needham & Wang Ling, *Science and Civilisation in China*, Vol. III: Mathematics and Sciences of the Heavens and the Earth, Cambridge, 1959, p. 10 n.

13 Johannes Bronkhorst, "A Note on Zero and the Numerical Place-Value System in Ancient India," *Asiatische Studien /Études Asiatiques,* 48.4 (1994) 1039-1042.

14 S. R. Sarma, "Śūnya in Piṅgala's Chandaḥsūtra," op. cit., pp. 126-136, esp. 132-133.

15 G. Coedès, "A propos de l'orgine des chiffres arabes." *Bulletin of the School of Oriental and African Studies*, 6 (1930-32) 323-328.

16 First published in *Epigraphia Indica* II, p. 19. For discussion, see Datta & Singh, op. cit., pp. 40-51. See also A. M. Shastri, "Mankhani Charter of Taralasvāmin and the Antiquity of Decimal Notation," *Annals of the Bhandarkar Oriental Research Institute,* 79 (1998) 160-177; Irfan Habib, "Joseph Needham and Indian Technology," *Indian Journal of History of Science,* 35 (2000) 245-

274, esp. 272-274; Georges Ifrah, *The Universal History of Numbers, From Prehistory to the Invention of the Computer,* translated from the French by David Bellos et al, New York, 2000, pp. 356-439, esp. p. 402 and figure 24.76.

17 Joseph Needham & Wang Ling, *Science and Civilisation in China*, Vol. III, Cambridge, 1959, pp. 10-11.

18 Takao Hayashi (ed. & tr), *The Bakhshālī Manuscript: An ancient Indian mathematical Treatise,* Groningen, 1995, esp. p. 89.

19 *The Patiganita of Sridharacarya, with an ancient Sanskrit Commentary,* edited by Kripa Shankar Shukla, Lucknow, 1959, Introduction, p. xxix.

20 A parallel situation obtains in the case of the expressions *nālikā* and *ghatikā*. The first expression meant originally an outflow water clock of cylindrical shape and time unit of 24 minutes measured by this clock. Around the fourth century AD, this type of water clock was replaced by the sinking bowl type of water clock which was called *ghaṭikā.* The time unit of 24 minutes measured by this variety of water clock was called *ghaṭikā.* But *nālikā* the older designation for this time unit also continued to be used. Often both are used as synonyms in the same text. In the South, however, *nālikā* survives in Tamil and Malayalam (with slight phonetic modification) while *ghaṭikā* (with slight phonetic change) survives in Kannada and Telugu.

21 S. N. Sen, "Transmission of Scientific Ideas between India and Foreign Countries in Ancient and Medieval Times," *Bulletin of the National Institute of Sciences of India,* No. 21 (1962) 8-30; idem, "Influence of Indian Science on Other Culture Areas," *Indian Journal of History of Science,* 5 (1970) 332-346; idem, "India and the Ancient World: Transmission of Scientific Ideas" in *The Cultural Heritage of India,* volume VI: Science and Technology, edited by Priyada Ranjan Roy & S. N. Sen, Calcutta, 1986 (reprint 1991), pp. 220-247; R. C. Gupta, "Indian Mathematics Abroad upto the Tenth Century A.D.," *Gaṇita-Bhāratī,* 4 (1982) 10-16; idem, "Sino-Indian Interaction and the Great Chinese Buddhist Astronomer-Mathematician I-Hsing (A.D. 683-727)," ibid, 11 (1989) 38-49. See also Wilhem Rau, *Indiens Beitrag zur Kultur der Menschheit,* Sitzungsberichte der Wissenschaftlichen Gesellschaft an der Johann Wolfgang Goethe-Universität Frankfurt am Main, Band III, Nr. 2, Wiesbaden, 1975.

22 Duan Yaoyong investigated this question in his master's thesis entitled "The Discussion that Indian Trigonometry affected the Chinese Calendar Calculation in Tang Dynasty (A.D. 618-907)",

Institute for the History of Science, Inner Mongolia Normal University, 1996, and came to the following conclusion: "I undertook a systematic study and organization of this topic for the first time (some made fragmentary study about it before); got that trigonometry knowledge from English translation of Indian Astronomy Corpus and tried to make it systematized (nobody did it before). I found an equivalent way that is equal to the method of trigonometry in China; made study of Chinese and Indian calendric calculation system, and tried to find the internal relations of them. I answer Dr. J. Needham's question [viz. 'How much trigonometry did the Tang astronomers know and where did they learn it?' with the reply that] 'Chinese calculation has nothing to do with Indian trigonometry. Chinese algorithm specialist solved the problem with 'the gougu theorem' which is equivalent way of trigonometry used by Indians,' trying to set out reasons why Chinese never accepted trigonometry." Cf. Thesis Abstract, *Gaṇita-Bhāratī: Bulletin of the Indian Society for History of Mathematics,* 20 (1998) 110-111.

23 *Alberuni's India,* tr. Edward C. Sachau, reprint: Delhi, 1964, vol. 2, p. 15.

24 S. N. Sen, "Transmission of Scientific Ideas between India and Foreign Countries in Ancient and Medieval Times," *Bulletin of the National Institute of Sciences of India,* No. 21 (1962) 8-30, esp. 24-25.

25 David Pingree, *Census of Exact Sciences in Sanskrit,* A-2, Philadelphia, 1971, s. v. Kanaka, p. 19.

26 In the fourteenth century Arab navigators report of their meeting the "Kanaka" on the Malabar coast, who must likewise be "Gaṇakas", i.e., astronomers or more specifically navigators. Personal communication from Dr Farid Benfeghoul, Institut für Geschichte der Arabisch-Islamischen Wissenschaften, Frankfurt.

27 See, inter alia, George Gheverghese Joseph, *The Crest of the Peacock,* pp. 311-12; see also Edgar Reich, "Ein Brief des Severus Sebokt" in: Menso Folkerts & Richard Lorch (ed), *Sic ad Astra: Studien zur Geschichte der Mathematik und Naturwissenschaften: Festschrift für den Arabisten Paul Kunitzsch zum 70. Geburtstag,* Wiesbaden, 2000, pp. 478-489.

28 "Algorizmi said: when I saw that Indians composed out of .IX. letters any number due to the position established by them, I desired to discover, God willing, what becomes of those letters to make it easier for the student ... Thus, they created .. IX. letters, figures of them are as follows ... The beginning of the order is on the right side of the writer, and this will be the first of them

consisting of unities. If instead of unity they wrote .X. and it stood in the second digit, and their figure was that of unity, they needed a figure of tens, similar to the figure of unity so that it became known from it that this was .X. Thus they put before it one digit and wrote in it a small circle "o", so that it would indicate that the place of unity is vacant." Cited in: B. A. Rosenfeld, "Al-Khwārizmī and Indian Science" in: W. H. Abdi et al (ed), *Interaction between Indian and Central Asian Science and Technology in Medieval Times,* Indian National Science Academy, New Delhi, 1990, vol. I, pp. 132-139, esp. 132.

29 See Charles Burnett, "Indian Numerals in the Mediterranean Basin in the Twelfth Century, with Special Reference to the 'Eastern Forms'" in: Yvonne Dold-Samplonius et al (ed), *From China to Paris: 2000 Years Transmission of Mathematical Ideas,* Stuttgart, 2002, pp. 237-288, esp. 239, 265-267.

30 Ibid.

31 This is no doubt an elegant solution for the nomenclature of the numerals of the form 1, 2, 3 etc. But how does one designate unambiguously the numerals associated with the Arabic-Persian script, such as ١٢٣ ? Again, when the Indian numerals were transmitted to Europe, they were first known as the "Indian" numerals. One wonders when Europe started calling them "Arabic" numerals and why.

32 S. R. Sarma. "Rule of Three and its Variations in India" in: Yvonne Dold-Samplonius et al (ed), *From China to Paris : 2000 Years Transmission of Mathematical Ideas,* Stuttgart, 2002, pp. 133-156.

33 *Āryabhaṭīya, Gaṇitapāda* 26.

34 *Vedāṅga Jyotiṣa of Lagadha in its Ṛk and Yajus Recensions,* with the Translation and Notes of T. S. Kuppanna Sastry, ed. by K. V. Sarma, New Delhi, 1985, pp. 40-41.

35 Joseph Needham & Wang Ling, *Science and Civilisation in China,* Vol. III, Cambridge, 1959, p. 146.

36 N. L. Maiti, "The Antiquity of Trairāśika in India," *Gaṇita-Bhāratī* 18 (1996) 1-8.

37 Lam Lay Yong, *A Critical Study of the Yang Suan Fa,* Singapore, 1977, p. 329: "The Rule of Three which originated among the Hindus is a device used by oriental merchants to secure results to certain numerical problems."

38 Adolf P. Juschkewitsch, *Geschichte der Mathematik im Mittelalter,* Leipzig, 1964, p. 204.

39 Ibid, p. 214. Al-Bīrūnī, *Fī Rāshikāt al-Hind*, in: *Rasá'ilu'l-Birúni by Abú Rayhan Muh. b. Ahmad al- Bírúní* ... The Osmania Oriental Publications Bureau, Hyderabad-Dn., 1948.

40 See, inter alia, D. E. Smith, *History of Mathematics*, II, p. 486 ff.

41 S. R. Sarma, "Perpetual Motion Machines and their Design in Ancient India," *PHYSIS: Rivista Internationale di Storia della Scienza*, Rome, 29.3 (1992) 665-676.

42 L. White Jr., "Tibet, India, and Malaya as Sources of Western Medieval Technology," *American Historical Review*, 65 (1960), pp. 515-526; reprinted in: idem, *Medieval Religion and Technology. Collected Essays*, Berkeley, 1978, pp. 43-57. See also, idem, *Medieval Technology and Social Change*, London, 1964, pp. 19-131.

43 A. Y. Al-Hassan, D. R. Hill, *Islamic Technology. An Illustrated History*, Cambridge-Paris, 1985, p.71.

44 Ibid., p. 70.

45 J. Needham, *Science and Civilization in China*, vol. IV, part 2, Cambridge, 1965, p. 540.

46 L. White Jr., *Medieval Religion*, op. cit., p. 53, n. 60 (= idem, *Medieval Technology*, op. cit., p. 130, n, 3).

47 For the text, see Bhāskara, *Siddhāntaśiromaṇi*, ed., by Bapu Deva Sastri, rev. by Ganpati Deva Sastri, Benares, 1920, Golādhyāya, Yantrādhyāya, vv. 50-53.

48 For the text, see *The Sūryasiddhnta*, with the Exposition of Ranganātha, the Gūḍhārtha-prakāśkā, ed. by F. Hall, reprint, Amsterdam, 1974, 13.16-18.

49 S. R. Sarma, "Perpetual Motion Machines and their Design in Ancient India," op. cit.

50 David Pingree, *Census of Exact Sciences in Sanskrit*, A 3, Philadelphia, 1971, pp. 40-41, s. v. Caṇḍīśvara; A 4, Philadephia, 1981, p. 368, s. v. Mallikārjuna Sūri.

51 In 1825, at Pondichery, one Swaminathan Seshayya showed John Warren how to compute the lunar eclipse of 31 May-1 June 1825 with the help of these *Vākyams*; cf. John Warren, *Kālasamkalita*, pp. 334-340: Fragment IV: "Computation of an Eclipse of the Moon by means of certain memorial and artificial words, and of shells in lieu of figures ... By Sami Naden Sashia, a Kalendar maker residing in Pondichery." In their study of this material collected by John Warren, Neugebauer and Van der Warden coined the phrase "Tamil Astronomy" which is actually "Kerala Astronomy".

52 S. R. Sarma, "On the Spread of the *Kaṭapayādi* System outside Kerala," forthcoming.

53 K. V. Sarma, "Word and Alphabetic Numerical Systems in India," in: A. K. Bag & S. R. Sarma (ed), *The Concept of Śūnya*, New Delhi, 2003, pp. 37-71, esp. 43-44.

54 K. V. Sarma, "Word and Alphabetic Numerical Systems in India," op. cit., p. 44.

55 Datta & Singh, *History of Hindu Mathematics*, Part I, pp. 69-72, assert that there were four variants of the *Kaṭapayādi* system, and this is repeated by everybody. In fact there were only two variants, viz. the standard system as practised in Kerala and the one employed by Āryabhaṭa II.

56 S. R. Sarma, *The Pūrvagaṇita of Āryabhaṭa's (II) Mahāsiddhānta*, Marburg 1966; see esp. Part I, pp. xx-xxii.

57 *Śiṣyadhīvṛddhidam of Lallācārya, with the Commentary Vivaraṇa by Śrīmad Bhāskarācārya*, ed. Chandra Bhanu Pandey, Sampurnanand Sanskrit University, Varanasi 1981.

58 On Rāmacandra Vājapeyin, see S. R. Sarma, "Astronomical Instruments in Mughal Miniatures," *Studien zur Indologie und Iranistik*, 1992, 16-17, pp. 235-276; David Pingree, *Census of Exact Sciences in Sankrit*, series A, Volume 5, Philadelphia, 1994, pp. 467-478; S. R. Sarma, "On the Life and Works of Rāmacandra Vājapeyin," forthcoming.

59 On Harṣa, see G. V. Devasthali, "Harṣa, the author of the Anka-yantra-cintāmaṇi and Relatives," *B. C. Law Volume*, part 1, Calcutta 1943, pp. 496-503.

60 Incidentally, this expression *Bhūtasaṃkhyā* appears to be a recent coinage; it is not attested in any Sanskrit text.

61 D. C. Sircar, *Indian Epigraphy*, Delhi, 1965, pp. 228-233.

62 G. H. Ojha, *Bhāratīya Prācīna Lipimālā*, p. 123; see also Datta & Singh, op. cit., pp. 70-71.

63 S. R. Sarma, "Kaṭapayādi Notation on a Sanskrit Astrolabe," *Indian Journal of History of Science* 34 (1999) 273-287.

64 S. R. Sarma, "Mathematical Literature in the Regional languages of India," forthcoming.

10

CALCULUS AND INFINITE SERIES IN SEVENTEENTH CENTURY EUROPE

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Other speakers at this Workshop have provided excellent insights into Kerala mathematics of the sixteenth and seventeenth centuries. In this paper I propose to outline advances and changes in European mathematics over the same period, to provide a basis for comparison so that we can more readily discern any links, connections, or parallel developments between southern India and western Europe. By the latter I mean essentially France, Italy, the Netherlands, and England, and I am going to discuss a core period of about one hundred and fifty years, from 1540 to 1690.

It is also perhaps necessary to define what we mean in this context by mathematics. Astronomical observations and calculations were carried out in medieval Europe as they were in medieval India for religious and astrological purposes. In Kerala such practices eventually led to the discovery of some sophisticated 'pure' mathematics, in particular the existence of certain infinite series. In Europe, on the other hand, they had largely fallen into decline by the sixteenth century and did not feature in the mathematical developments of the seventeenth century. They will therefore not be discussed in this paper. Nor do I intend to treat the everyday mathematics of trade and

commerce, since our concern here is with what we may call 'higher' mathematics, as developed and studied by small intellectual elites, and passed on through personal teaching and extended periods of disciplined study.

At the beginning of the period in question, around 1540, European mathematics had advanced very little since the middle ages. By the end of it, a century and a half later, mathematicians in western Europe had developed an entirely symbolic and algebraic mode of writing and were in possession of the fully fledged calculus: algorithmic methods for finding tangents to curves, and areas of volumes of curved spaces, together with a clear understanding that the two processes were fundamentally related. Further they had infinite series for a wide range of algebraic and trigonometric functions (though the word 'function' itself was not yet in use). How this all came about, as I understand it, is what I will describe in what follows.

1. Greek And Hellenistic Influence

It is already well known that several major strands of thought from outside western Europe played a vital part in the story, one of the foremost being the legacy of mathematics from the Greek-speaking world of the Classical and Hellenistic periods. After the collapse of the Roman Empire in the west, little more survived of this than some practical geometry based on land measurement and a few elementary scraps of Euclid. From as early as the mid-twelfth century onwards, however, Euclid became much better known through new translations from Arabic, made by scholars from more northern parts of Europe who travelled to Spain, Sicily, or north Africa. The early translations were not full versions of the text: sometimes they contained only the easiest propositions, or gave theorems without proof, thus failing to convey Euclid's careful deductive structure of definitions, axioms, and propositions. At this stage there were also translations from Arabic to Latin of Ptolemy's *Almagest*, and of some of the treatises of Archimedes.

During the sixteenth century, the situation changed dramatically in mathematics as in all other intellectual disciplines. Greek, which for centuries had been virtually lost in

northern Europe, was rediscovered as a language of learning and culture. The rediscovery of Greek manuscripts from Byzantium and elsewhere, and the ability of scholars to read and translate them, meant that Latin translations could now be based directly on Greek originals, and printing allowed classical texts to be made available outside the confines of monasteries or universities. One of the foremost translators of Greek texts in the mid-sixteenth century was Federico Commandino, who over a period of thirty years from 1558 onwards produced new Latin editions of Euclid, Archimedes, Apollonius, Ptolemy, Pappus, and others, working directly from manuscripts available in the Vatican library in Rome and elsewhere. A manuscript copy of Diophantus' *Arithmetica*, also in the Vatican library, was translated and published by Wilhelm Holzman (Xylander) in 1575.

The impact of these works on European mathematicians cannot be overestimated. Rafael Bombelli (*c.* 1526–1572), for instance, had already completed the third book of his *Algebra* when the Vatican manuscript of Diophantus' *Arithmetica* came to his attention, and he immediately abandoned what he had written to take into account this new material. Greek geometry made a similar impression: to mathematicians who had grown up in the relative poverty of the medieval curriculum, the richness and depth of Euclid, Archimedes, or Apollonius must have seemed quite astonishing, and it is little wonder that Greek geometry came to be seen as the apotheosis of mathematical achievement, to which modern Europeans might aspire but could hardly surpass.

The situation was also, however, in many ways frustrating: too many texts were incomplete, damaged, or lost. The *Collections* of Pappus (*c.* 300 AD) provided an invaluable source of information on what had once existed, but also showed just how much was lost. In particular, of the eight books of *Conics* of Apollonius, only the first four were known in Greek in the sixteenth-century, and Pappus gave only hints as to what had existed in the other four. There were mathematical difficulties too: the standard Greek method of proof by *reductio ad absurdum*, in which a result was shown to be true by demonstrating that its negative led to a contradiction, provided

admirable rigour, but gave no insight into how the theorem had been discovered in the first place. Thus Europeans found a treasure trove of new results, but at the same saw that much work needed to be done to recover what was lost, or to complete or elucidate what existed.

2. Islamic Influence

The second major strand in the revitalization of European mathematics came from the Islamic middle East, and was never as obviously or as ostentatiously important as the recovery of Greek texts, yet in the end it was to be equally significant. Al-Khwārizmī, working in Baghdad early in the ninth century, wrote two elementary texts, both of which proved to be seminal: one on Indian numerals and associated methods of calculation; the other on the equation-solving techniques of *al-jabr wa'l muqābala*, restitution and balancing. For the next four centuries or more, all further texts on these two subjects were essentially based on al-Khwārizmī's. I do not propose here to discuss the dissemination and use of Indian numerals, except to say that by the sixteenth century they were well established throughout Europe for astronomical and mathematical purposes, although the older Roman system also remained in everyday use. The spread of *al-jabr*, or algebra, though for a long time less visible than that of the Hindu-Arabic numerals, was eventually to be crucial to the development of European mathematics.

Al-Khwārizmī's *Al-jabr wa'l muqābala* (*c.* 825 AD) contains instructions on how to handle six classes of quadratic equation (squares equal to numbers, squares plus roots equal to numbers, and so on). The methods, and their geometric underpinning (a literal 'completion of the square'), reveals close resemblance to problems and solutions given in Old Babylonian texts *c.* 1800 BC, but how or why they re-emerged in ninth-century Baghdad is not known. From 825 AD onwards, however, the methods were firmly established as part of the mathematical canon. At about the same time as Euclid was first translated from Arabic to Latin, the *Al-jabr* was also translated into Latin, also in Spain, and by two of the same scholars from northern Europe: Robert of Chester (*c.* 1145) and Gerard of Cremona (*c.* 1175).

Meanwhile material from the *Al-jabr* or from later texts which were based on it, such as those of Abū-Kāmil (*c.* 850–930) or al-Karajī (*c.* 1010), was taken up by Leonardo of Pisa (Fibonacci) in his *Liber abaci* (1202), and became absorbed into the abacus teachings of thirteenth- and fourteenth-century Italy. From there it slowly spread northwards into Germany and France, and by the sixteenth century even to England.

It was in northern Italy in the sixteenth century that the first real advances were made beyond the methods for quadratic equations taught by al-Khwārizmī. Italian mathematicians at last found correct and reliable general methods for solving cubic and biquadratic equations, and these were published by Girolamo Cardano in the *Ars magna* in 1545, right at the beginning of the period on which I have chosen to focus. So important was this breakthrough that Cardano's text set the agenda for the European development of algebra for the next 60 or 70 years.

I must emphasize here that I am using the word 'algebra' in the way that sixteenth-century European practitioners themselves used it: for the branch of mathematics concerned with the solution of second, third, or fourth degree equations. I am *not* referring to the related but somewhat different process of using special words, abbreviations, or symbols to stand for numbers. In western mathematics this process first appears in a rudimentary way in the *Arithmetica* of Diophantus, and later in the *cossist* algebra of the sixteenth century (where the abbreviation *co.* was frequently used for *cos*, the unknown *thing*, alongside *cu.* for cube, and so on). A fully symbolic notation was developed only around 1600.

3. A New Synthesis

It was in the work of François Viète (1540–1603) that the two major forces behind European mathematics, the Greek and Islamic, finally came together. Viète was profoundly influenced by his reading of Greek texts, and introduced or re-introduced many Greek terms into his writing. In particular he was keen to develop Pappus' concept of 'analysis' as a method of discovering or confirming results. But Viète also knew the algebra of Cardano, and saw what no European mathematician

had previously seen, that algebra could be applied not just to numerical problems but to geometry itself. Thus algebraic analysis, a fusion of Greek analysis and Islamic algebra, became a new and powerful method, by which Viète hoped to recover the lost results of the ancient Greeks and to solve problems that had previously appeared intractable. Viète did not live long enough to develop or publish his work in full, but similar ideas were taken up and developed by others: by Thomas Harriot (*c.* 1560–1621) in England, Pierre de Fermat (1601–1665) in France, and René Descartes (1596–1650) in the Netherlands. The resulting fusion of algebra and geometry was fundamentally crucial to the later development of calculus in Europe.

One further important advance, already briefly referred to above, should not be overlooked, and that is the development of modern algebraic notation, first by Harriot, in England and later by Descartes in France. Again the significance of this can hardly be overestimated: algebraic notation not only makes mathematical results easier to write and to communicate, but it also enhances clarity and speed of thinking and understanding. In other words, it is not just a convenience, but a mathematical tool of major significance.

So much then for the background influences and achievements that had helped to form the mathematics of north-west Europe by the start of the seventeenth century. Now we turn to look at some of the particular problems, and attempts at solutions, that inspired the calculus.

4. The Development Of The Calculus

(i) Tangent problems

A class of problems that later came to be handled with consummate ease by the calculus was that of finding tangents to curves. Here the basic principle were to be found in the *Conics* of Apollonius, but only for a restricted class of curves, the conic sections. The problem was taken up in particular by a number of French mathematicians of the early seventeenth century: Fermat, Descartes, and Roberval.

Fermat devised his tangent method as early as 1629 when he was still a law student at Bordeaux, and for him the problem sprang directly from his study of Greek mathematics and the work of Viète. In fact his tangent method arose from a closely related problem, of maxima and minima. Fermat not only solved the maxima and minima problem but immediately extended the method to find the general tangent to a parabola. His description of the method abounds with references to Greek writers: Archimedes, Apollonius, Diophantus, and in many ways his style of writing and his method of working rested on theirs. In one important respect, however, he broke away from the Greeks, and introduced into European mathematics something that was to give it enormous power and yet was to be the source of long-running mistrust and scepticism: the infinitely small quantity. Fermat himself did not describe it in such a way, but simply denoted it by the letter E (because it was a second unknown quantity, after A). In the process of finding his tangent he first needed to divide by E (in modern notation, to reduce $DE^2 + A^2E = 2DAE$ to $DE + A^2 = 2DA$), and then let E vanish (to reduce the last statement to $A^2 = 2DA$, or $A = 2D$). Fermat's answer was correct, as he knew from his reading of Apollonius, but the method involves dividing by a quantity that is essentially non-existent. The difficulties did not go unnoticed, and problems about the nature and existence of infinitely small or vanishing quantities, and how they should be handled, remained a fundamental difficulty for the next two centuries.

Descartes devised a quite different method of tangents that avoided any use of the infinitely small. He saw that it was possible to construct a normal to a curve by finding what is essentially its radius of curvature. This he did by finding a circle with only a single point of intersection with the curve at a given point. The radius of the circle is normal to the curve, and from the normal it is easy to construct the tangent. The idea is simple, but in practice the algebraic manipulation rapidly becomes too difficult for comfort. Descartes, however, was rightly proud of the method, and published it in *La géométrie* in 1637.

A third way of exploring tangents, used by Giles Personne de Roberval (1602–1675) in the late 1630s, and a little later also by Evangelista Torricelli (1608–1647), was to consider a curve

as the path of a moving point, with the tangent representing the direction of motion.

Later René Sluse (1622–1685) and Johann Hudde (1629–1704) discovered useful algebraic rules, but these were based essentially on insights similar to those of Fermat and Descartes.

(ii) Quadrature

More urgently pursued, and over a longer period of time than tangent questions were problems of quadrature: finding areas bounded by curves (together with related problems of cubature, or volume). Here the initiatives were more scattered and more widespread, both geographically and over time. As early as 1615 Johannes Kepler (1571–1630) investigated the volume of solids by considering them as sums of infinitesimal parts (a sphere as a sum of cones, for instance). The main theoretical advances, however, came with the theory of indivisibles, developed independently by Grégoire Saint-Vincent (1584–1667) in the low countries, by Roberval in Paris, and by Bonaventura Cavalieri (1598–1647) in Bologna, all in the 1620s. Roberval never published, and Saint-Vincent's *Opus geometricum* did not appear until 1647, so the most influential of the three was Cavalieri, whose *Geometria indivisibilibus* was published in 1635.

For Cavalieri an indivisible was a line cutting a surface, or a surface cutting a solid, and in principle, he argued, one could compare areas or volumes by comparing the indivisibles from which they were constituted. This was essentially a method of comparison rather than of finding absolute values, but some of his disciples and followers, particularly Torricelli in Italy and John Wallis (1616–1703) in England, simplified the theory to the point of arguing that an area *is* the sum of its lines, and a volume *is* the sum of its constituent surfaces. But here again there arises a problem similar to that we have already observed for Fermat, the problem of infinitely small or indivisible quantities. Does an indivisible have some discernible thickness or not? In principle it does not, since by definition it is indivisible; but if it has none, then even by taking infinitely many of them one can never fill a plane or solid of finite area or volume.

Seventeenth-century writers, excited and inspired by the results they could achieve, largely ignored such difficulties, and Wallis in particular put the method to good use to find the result we would write in modern notation as $\int_0^X x^n dx = \frac{X^{n+1}}{n+1}$, for n positive, negative, or fractional. (Saint-Vincent, Fermat, Roberval, and Cavalieri had already been able to deal with integer values.) Encouraged by his success in these relatively simple cases Wallis went on to try to attempt the quadrature of the circle, another problem that had come down to Europeans from Greek writers. Wallis could not handle the problem by his previous methods because he had no way of expanding $(1 - x^2)^{1/2}$, so he fell back on ingenious methods of numerical interpolation. The second half of his *Arithmetica infinitorum* (1656) is filled with one numerical table after another as Wallis gradually closed in on his objective, finally delivering his famous infinite fraction for $4/\pi$.

(iii) The calculus of Newton

In the winter of 1664–65 Isaac Newton (1642–1727) took refuge from the plague in Cambridge at his home in Lincolnshire. His manuscripts from that period were published in the late 1960s, in the first volume of D T Whiteside's comprehensive eight-volume series of Newton's mathematical papers, and so the evidence of his thought in the next two extraordinary years is now fully available to scholars. It is clear from his manuscripts that Newton arrived at an understanding of the basic principles of the calculus not by any simple linear route, but by working simultaneously on a number of different ideas which he was able to combine. His work on quadrature, for instance, developed directly from his reading of Wallis's *Arithmetica infinitorum*, and was closely connected with his discoveries of infinite series, as will be seen below.

Elements of several earlier methods can be discerned in Newton's method of tangents: he acknowledged his debt to Descartes' *La géométrie*, and also knew of Hudde's rules, published in 1659, for finding repeated roots and maxima and minima. He could not have known directly of either Fermat's

work or Roberval's, which as yet circulated only in manuscript, though Newton's tangent method bore many similarities to both, and to Fermat's in particular.

Within a year or little more, Newton was able to combine his results on quadrature with his results for tangents, and to recognize the inverse nature of the relationship between them. Up to this point Newton had certainly drawn on the published work of others, but the credit for the crucial breakthrough to what later came to be called the Fundamental Theorem of Calculus must go to Newton alone: still only in his early twenties, he worked at this period in almost total isolation.

Newton faced the same difficulties as his predecessors with infinitely small or vanishing quantities, and attempted to overcome them by developing the idea of 'prime and ultimate ratios', the ratio of two quantities in the first or last moment of their existence. As is clear from this, he had come to envisage the calculus in terms of geometric motion and change, and his mature calculus was described in the language of fluents (flowing quantities) and fluxions (rates of change).

Newton did not immediately publish or reveal his results, until prompted to do so by the publication of Nicolaus Mercator's *Logarithmotechnia* (see below) in 1668. In 1669 he wrote a treatise entitled *De analysi per aequationes numero terminorum infinitas*, which he sent to John Collins in London and Isaac Barrow in Cambridge, and in 1670 began to prepare a longer treatise on fluxions and infinite series, but unwelcome controversy over his work on optics caused him to withdraw it. Neither treatise was published until the early eighteenth century. The first publication of his calculus by Newton himself was as an appendix to his *Opticks* in 1704.

(iv) The calculus of Leibniz

During the years 1673 to 1676, Gottfried Leibniz (1646–1716), then in his early twenties, also began to make discoveries that would lead to the calculus, but for Leibniz inspiration came primarily from continental sources. Under the guidance of Christiaan Huygens in Paris, he studied the work of Cavalieri, and of Blaise Pascal (1623–1662), and developed Cavalieri's idea of *omnes lineae*, 'all the lines' in a figure, and some of Pascal's results on sums of differences. Beyond that Leibniz, like Newton, laboured very much alone, and, as for Newton, his surviving manuscripts reveal in detail the slow and uneven progress of his thought. Building on the simple concept of differences and sums as inverse relationships, Leibniz came to recognize, just as Newton had, the inverse relationship between tangent and quadrature. He soon began to represent a small difference, or increment, by the notation dx, and he abbreviated *omnes lineae* first to *omn. lin. y* and eventually to $\int ydx$. Leibniz published his first account of the differential calculus in the *Acta eruditorum* in 1684, followed by a second paper on the integral calculus in 1686.

Leibniz's calculus leads, of course, to exactly the same results as Newton's, but the original discovery and formulation were completely different. Concepts of change and motion which were so much part of Newton's thought were absent from Leibniz's calculus, which was altogether more formal and algebraic. Leibniz, like Newton, at first had to grapple with infinitesimally small or vanishing quantities, without being either clear or explicit about what they were or how they should behave, but once he had become confident with his method, he was able to avoid such discussions and present his calculus simply as a set of formal rules (showing what one must do to differentiate a product, or quotient, for example). This rule-based approach made Leibniz's calculus much easier to learn than Newton's, and the fact that Leibniz also published first meant that it was his version and his notation that became most rapidly accepted, especially outside Britain.

To summarize this section, both Newton and Leibniz discovered the fundamental theorem of calculus, quite independently and yet within about eight years of each other. In each case, thanks to the prolific quantity of surviving manuscripts, we can trace the development of their thoughts in minute detail. The sources that they drew on and their approaches to the subject matter were very different. Newton was influenced primarily by the Cartesians and by Wallis; Leibniz by Cavalieri, Pascal and Huygens. Newton conceived his calculus in terms of fluxions and vanishing quantities; Leibniz as

a set of formal algebraic rules. It is clear that many of the necessary steps towards the calculus had already been taken or foreshadowed earlier in the century by mathematicians working as far apart as Germany (Kepler), France (Fermat, Roberval), Italy (Cavalieri, Torricelli), the low countries (Saint-Vincent, Descartes, Hudde), and England (Wallis), to mention only the most important of them. Nevertheless, the full discovery of the calculus was eventually a *tour de force* of the lone genius of two different but very remarkable minds.

5. Infinite Series

(i) Newton's infinite series

The emergence of infinite series in European mathematics was intimately bound up with the discovery of the calculus, and occurred at much the same time, from the early 1660s onwards. The first published series appeared in the *Logarithmotechnia* of Nicolaus Mercator (1620–1687) in 1668. The series and Mercator's derivation of it were simple: by straightforward long division of the kind that had been used for centuries for numbers, he found that $\frac{1}{1+a} = 1 - a + a^2 - a^3 + \ldots$. It was the publication of the *Logarithmotechnia* that prompted Newton to reveal that he too had discovered such a series, though initially by different methods. In fact during that productive winter of 1664–65 Newton had gone already very much further than Mercator.

Newton later wrote to Leibniz in 1676 that he had found infinite series by no fewer than three different methods: (*a*) by numerical interpolation; (*b*) by algebraic operations (such as long division and root extraction); and (*c*) by solving equations. The manuscript evidence confirms that Newton's claim was essentially correct, and we will look at each of these three methods briefly in turn, because they also lie at the heart of the other and later series discoveries by European mathematicians.

(*a*) From the manuscript notes that Newton made as he read Wallis's *Arithmetica infinitorum*, we can see how, as he came to the end of the book, he continued in an unbroken way with the same train of thought, and yet took it to new levels. Just as Wallis had done, Newton investigated areas by methods of numerical interpolation, but he made two significant advances. First, he found a more sophisticated and more easily applicable method of interpolation, allowing him not only to check and confirm Wallis's results, but also to produce very many new results of his own. Second, he introduced a variable quantity, called x, which enabled him to express his findings as a series of powers of x, rather than numerically as Wallis had.

Thus, Newton was able to produce his first two infinite series, for $(1 + x)^{-1}$ and for $(1 - x^2)^{½}$, by purely numerical interpolation arising directly from his study of Wallis's methods. But further, from the sheer quantity of additional results he discovered, he was soon able to deduce the formulae for the coefficients in the general binomial expansion, that is, coefficients of powers of x in the expansion of $(1 + x)^{p/q}$. This enabled him to progress to infinite series of all kinds, not just for algebraic expressions, but for quantities such as sin, arcsin, log, and antilog. And once Newton had found such series he could integrate them term by term to find quadratures of the relevant curves.

(*b*) When he wrote *De analysi* in 1669 Newton also experimented with other methods of finding series, amongst them Mercator's method of long division. Similarly, the series for $(1 - x)^{½}$ can be found by generalizing the traditional numerical algorithm for root extraction, and it is easy to check by multiplication (as Newton indeed did) that the result is correct. Newton also knew how to invert series, so that the series for arcsin, for example, was converted relatively easily into the inverse series, for sine.

(*c*) The third and final method mentioned by Newton also appears in *De analysi*. It is a method of solving equations by successive approximations, so that the solution equation appears as an infinite series of powers.

With later refinements the process came to be known as the Newton-Raphson method.

(ii) Leibniz's infinite series

As already observed, the discovery of infinite series was completely inseparable from the discovery of the calculus, and Leibniz, like Newton, also began to discover such series as his calculus took shape. (In fact through his correspondence with Henry Oldenburg and John Collins in London, Leibniz was aware that Newton had already discovered certain series, but did not know at first what they were.)

For Leibniz the breakthrough came through his transmutation theorem, in which he saw that the infinitesimal strips that fill a certain area, could be replaced by infinitesimal triangles radiating from a common origin. The relationship between the two is simple to work out in geometric terms. It is in fact a special case of what later came to be called integration by parts, and one of the crucial features of it is that an expression representing an area is seen to be directly dependent on the tangent to the same curve. Leibniz used his transmutation theorem in particular to find the area of a quadrant. After a sequence of transformations and manipulations he needed to integrate the function $\frac{z^2}{1+z^2}$, and used long division to write it as $z^2(1 - z^2 + z^4 - z^6 + \ldots)$, which he could integrate term by term to produce $\frac{z^3}{3}-\frac{z^5}{5}+\frac{z^7}{7}-\frac{z^9}{9}+\ldots$. This then gave him the area of a quadrant as

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

Unlike Newton's series, Leibniz's discovery is numerical, not algebraic. By the end of 1676, however, he learned more about Newton's series, from the famous *epistola prior* and *epistola posterior* written to him by Newton in June and October that year. And as the content of Newton's letters began to be more widely known (long extracts of both were printed by

Wallis in his *Treatise of algebra* in 1685), infinite series began to pass into more general use.

(iii) Gregory's infinite series

In an account of infinite series there is one other mathematician we must discuss in some detail, and that is James Gregory (1638–1675), who at the end of his all too short life was Professor of mathematics in Edinburgh. Gregory was unusual amongst British mathematicians in having studied abroad: from 1664 to 1667 he lived in Padua and studied under Stefano degli Angeli (1623–1697) of the Gesuati order. Angeli had been a pupil of Cavalieri, also of the Gesuati, until the death of the latter in Bologna in 1647, and later referred to Cavalieri as the 'Herculean column of Italian geometers'. During his years in Padua, Gregory wrote two treatises, *Vera circuli et hyperbolae quadratura* (1667) and *Geometriae pars universalis* (1668), on problems of quadrature and centres of gravity, both clearly influenced by Angeli. On his return to Scotland he published a short third treatise, his *Exercitationes geometricae* (1668), containing a miscellany of results. Among them are the relationships we would now write as $\int \tan x dx = \log\sec x$, and $\int \sec x dx = \log(\sec x - \tan x)$, both proved by geometric procedures, including methods borrowed from Grégoire Saint-Vincent's *Opus geometricum* (1647).

Only in 1670 did Gregory offer some hints, in letters to John Collins, that he had been working on infinite series. His results were no less impressive than Newton's, but the evidence as to how he arrived at them is very much scarcer, and correspondingly more difficult to interpret or evaluate. Gregory was aware from Collins that Newton had achieved some similar results and, believing that Newton had priority, abstained from publication. Unfortunately Gregory died in 1673, at the age of 36, long before Newton had published anything, and left behind no more than a few letters and jottings from which we must try to glean his thoughts.

Amongst Gregory's papers are some notes on the back of a letter, concerning some of the series that he sent to Collins a few

days later. The letter (from an Edinburgh bookseller) was dated 29 January 1671, and Gregory wrote to Collins on 15 February. It is quite clear that the notes and the letter to Collins are related, since both contain an identical arithmetical error. The notes are too sketchy, however, to constitute Gregory's entire work on the subject, and look more like a reminder to himself, or a hasty re-working of something he had already developed more fully earlier. The notes essentially show how, given an ordinate at one point on a curve, it is possible to calculate the ordinate at a nearby point by successive approximation. This is very close to the method that Newton had developed for solving equations (the forerunner to the Newton-Raphson method), but where Newton had applied it to algebraic equations, Gregory now applied it to trigonometric functions.

Gregory's notes contain two very important ideas. First they show that he knew how to calculate rates of change for trigonometric quantities: sine, tangent, secant, and so on. Some of this is clear already in his *Exercitationes* of 1668 where he had worked with some of these ideas geometrically, but by 1671 he was able to write the relationships in concise algebraic notation. The twentieth-century editor of Gregory's papers, H W Turnbull claims that Gregory was 'differentiating', but this is to attribute to him a process that had not yet been developed or named, at least not publicly. Nevertheless Gregory's grasp of these trigonometric relationships is remarkable. So is the second feature of his work, which would be described in modern terminology as the ability to differentiate a compound function. It is difficult to describe Gregory's insight in anything other than modern notation, in which it becomes

$$\frac{df(q)}{dx} = \frac{df(q)}{dq}.\frac{dq}{d\theta}.\frac{d\theta}{dx} \qquad (*)$$

For one of the particular examples chosen by Gregory, the quantity we have written here as $\frac{dq}{d\theta}.\frac{d\theta}{dx}$ remains constant as $q+\frac{r^2}{q}$, and one sees Gregory successively 'differentiating' with

respect to q, and then multiplying by $q+\frac{r^2}{q}$, just as the formula (*) requires. Occasional slips in the working help to reveal his thought processes and confirm that he was indeed following such a procedure. Once again this was a remarkable and sophisticated process that allowed Gregory to develop infinite series for such functions as arctan θ (which readily gives Leibniz's series on substituting $\pi/4$ for θ), sec θ, log tan θ, and others. It is a great loss that we do not have more of Gregory's working, for there is no doubt that he had arrived at some extremely valuable insights.

In conclusion to this section we may say that by the 1660s, and particularly in Britain, infinite series, like the calculus itself, were 'in the air', and began to emerge in the writings of various authors, and in various ways. In general Newton's summary of his three methods covers the possible approaches, though only he, following Wallis, made significant use of method (*a*), numerical interpolation. Meanwhile, Mercator, Newton, Gregory, and later Leibniz, all made use of method (*b*), algebraic manipulation, to find series for reciprocals, while Newton used it more extensively, for the extraction of roots and for inversion of series. The most sophisticated of Newton's three methods was (*c*), that of successive approximation. In Newton's hands this led to the process that was later called the Newton-Raphson method for numerical solution of algebraic equations. Gregory seems to have applied a similar process but in quite different circumstances to ordinates of trigonometric curves: he had somehow worked out the required rates of change for trigonometric functions, and was able to apply them to compound quantities using something very close to what is now called the 'chain rule'.

For Newton and for Leibniz we have detailed manuscript evidence of their working, but for Gregory, who made such remarkable advances, we have tantalisingly little.

6. Conclusion

I have tried to show in this paper how European mathematics of the late sixteenth century was profoundly influenced both by the classical Greek legacy and by later contributions from the

Islamic middle East. It was the powerful fusion of Greek analysis and Islamic algebra that eventually laid the groundwork for the eventual emergence of the calculus in the hands of Newton and Leibniz in the second half of the seventeenth century. The problems that led most directly to the calculus were classical problems of quadrature and tangents, or of maxima and minima, arising from the geometric study of curves. The methods used to solve them, however, came to depend increasingly on algebraic formulations, both in the analyses and the solutions.

For Newton and Leibniz the emergence of infinite series was inseparable from that of the calculus itself. Important advances were also made by James Gregory, but since he did not publish in his lifetime, his work was not immediately influential. The surviving manuscripts of Newton and Leibniz give us detailed insight into their sources and methods, and enable us to trace in considerable detail the influence of their predecessors and contemporaries, as well as the unique personal contributions made by each of them.

DISCUSSION

The period covered by this paper more or less coincides with the period of Jesuit presence in the Malabar. The paper began with the view that a geographical as well as historical context should be taken into account when discussing a major intellectual development that flourished both in India and Europe. In Europe, the infinite series was deployed in three different ways - infinite series by interpolation, by division and extraction of square roots and by solving equations. There may have been a fourth method as well, since Gregory in his derivation of the arctan series used what may well have been a geometric method whose details are, however, not known to us. In Kerala as well, there were also different approaches, although the geometric approach achieved a greater degree of prominence. The paper argues that a comparison between the Indian and European methods would indicate independent and parallel developments and where there were striking similarities, they were arrived at by very different approaches.

It was, however, pointed out that without the Indian number system, the infinite series could not have been conceived. Further, there was the question as to where and when did the idea of dealing with infinity come from into Western Europe? There is little to suggest that the source of this idea was to be found in Hellenistic mathematics. And if European mathematicians were aware of later developments in Islamic mathematics which dealt implicitly with series of infinite terms then they were capable of not only being able to appreciate mathematics within a different epistemology but also in a different scripted language. A search for an Indian connection among some of the key figures in European mathematics has not as yet yielded any supporting evidence.

By arguing that modern calculus developed as the outcome of uniting the methods of Arabic algebra and Greek geometry to deal with problems of the tangent and quadrature originally rooted in Greek science, the paper suggests that the emergence of modern calculus in Europe occurred without any Indian influence. And also, as the paper argues, even if there had been an Indian influence it did not have played any necessary role since the European calculus could well have developed without it. However, this view fails to answer why modern calculus did not develop in the Arab world since both algebra and Greek geometry were also known there, and the problem of tangents and quadrature addressed by the ancient Greeks were also discussed and even extended by Arab mathematicians.

The paper fails to address the documentation by others reported elsewhere in this Proceedings who have found strong circumstantial evidence supporting the influence of Kerala mathematics on calculus in Europe. Its failure to address and rebut these claims bring into question the assertion of parallel independent developments in the two mathematical traditions. It was suggested during the discussion that the value of the paper could have been higher and the insights offered deeper had it included a comparison of the methods of the European and Kerala schools. This was an expectation since at the outset of this paper, the author declares: "In this paper I propose to outline advances and changes in European mathematics over the same period, to provide a basis for comparison..."

It was also argued that the paper underestimates the significance of the number system, independent of any influence from Kerala mathematics, which made possible the development of infinite series and infinitesimals in mathematics. The infinite series itself is a generalization of the representation of finite numbers as a series based on powers of ten in the Indian decimal system. , Hence, the underlying philosophical orientation may have had an Indian influence, i.e. the Indians working within a system where numbers could be written in the form of an infinite power series could through the effort of Madhava make a 'passage to infinity' whereas such a passage would have been very difficult in Europe which had only its Greek and Islamic heritage to build on. The question remains: how did the European mathematicians make the magic jump from the finite to the infinite

To the question whether the infinite series discovered in Europe similar to the Indian ones, the author of the paper pointed out that while the origin of series for π was different for the two mathematical traditions, there could have been only one series for sine and cosine and the inverse tangent, the series were identical. In any case, their use for computation was strictly limited. However, the Kerala mathematicians devoted considerable attention to the introduction of partial correction factors and transformations to get over the problems of slowly converging series. These procedures, taken in conjunction with the original series, would have facilitated the construction of 'rules of the thumb' for 'practitioners' (such as navigators and craftsmen) to evaluate accurately the values of sines, cosines and circumferences for given diameters.

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PART – V

The Jesuit Conduit : The Background and Evidence

11

A REPORT ON THE INVESTIGATION ON THE POSSIBILITY OF THE TRANSMISSION OF THE MEDIEVAL KERALA MATHEMATICS TO EUROPE

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INTRODUCTION

The longstanding belief that the calculus was invented independently by Newton and Leibniz in the late 17th century[1] after exploiting the works of European pioneers such as Fermat, Roberval, Taylor, Gregory, Pascal, and Bernoulli[2] in the preceding half century had originally been refuted by Charles Whish in his seminal paper in 1835. Since then it has became well known that the fundamental elements of the calculus including numerical integration methods and infinite series derivations for π and for trigonometric functions such as $\sin x$, $\cos x$ and $\tan^{-1} x$ (the so-called Gregory series) had already been discovered over 250 years earlier in Kerala. These developments first occurred in the works of the Kerala mathematician Madhava and were subsequently elaborated on by his followers Nilakantha Somayaji, Jyesthadeva, Sankara Variyar and others between the 14th and 16th centuries.[3] There are several modern journal papers[4]

and European histories of mathematics[5] which acknowledge the work of the Kerala School.

It is relevant in the context of the findings of the investigation of the transmission conjecture to note that Kerala pre-calculus was founded on an epistemological base that was different from that of the Greek epistemological base predominant in Europe. The Kerala mathematics appears to have evolved from a long line of mathematical developments in India under the epistemology expounded by Aryabhata I in his book *Aryabhatiya* composed in AD 499. Here the ideas of mathematics and proof differ from the Greek Platonic or modern Hilbertian idea that mathematics must be divorced from the empirical. A broader discussion of the differences in epistemology will be given by other speakers in the workshop. This paper will focus on the investigation of the transmission conjecture.

A basis for establishing the transmission of knowledge between communities incontrovertibly is, of course, *direct* evidence of translations of the relevant manuscripts, their understanding by scholars of the donor community, and the use of the transmitted knowledge (with or without further developing the ideas). The transmission of Indian mathematics and astronomy since the early centuries AD via Islamic scholars to Europe has been established by direct evidence. The transmission of Indian computational techniques was in place by at least the early 7th century for by 662 AD it had reached the Euphrates region[6]. A general treatise on the transmission of Indian computational techniques to Europe is given by Benedict.[7] Indian Astronomy was transmitted westwards to Iraq, by a translation into Arabic of the *Siddhantas* around 760 AD[8] and into Spain. This transmission was not just westwards for there is documentary evidence of Indian mathematical manuscripts being found and translated in China, Thailand, Indonesia and other south-east Asian regions from the 7th century onwards.[9]

However, in the absence of direct evidence some weaker paradigms are considered by some historians to be sufficient. For example Neugebauer[10] considered that establishing the following criteria to be sufficient to establish transmission:

- the identification of methodological similarities
- the existence of communication routes
- a suitable chronology for the transmission.

Neugebauer used his paradigm to establish his conjecture about the Greek origins of the astronomy contained in the Indian astronomical treatises in the *Siddhantas*. A much weaker paradigm is van der Waerden's 'hypothesis of a common origin'[11]. Essentially van der Waerden used the fact of chronological priority and the implicit inherent 'superiority' of Greek science to conclude that various mathematical developments, especially those in India, must have been transmitted from Greece. For example, van der Waerden used the 'hypothesis of a common origin' to claim that Aryabhata's trigonometry[12] was borrowed from the Greeks. He made a similar claim about Bhaskara II's work on Diophantine equations. These claims were stated in one of his works[13] in which he further states that "... in the history of science independent inventions are exceptions: the general rule is dependence"[14].

If the community of historians of mathematics were to be convinced of the soundness of van der Waerden's paradigm then the chronological priority of the Kerala mathematics would clearly have been sufficient to establish our transmission conjecture. However, we find both the van der Waerden and Neugebauer paradigm problematic for two reasons. The first, the less important one, is that there is a Eurocentric attitude that even persisted into the late 20th century (and may persist even now) which will militate against the use of this paradigm in the case at hand: being traces left of the concerted attempts made at the end of 19th century by some prominent European historians of mathematics such as the Sedillot to discredit and downgrade the science of the European colonies[15]. Further, even into the mid 20th century, there was an explicit disregard of the later developments in Indian mathematics as signposted by Charles Whish as early as 1835. This is what an influential historian of mathematics, D E Smith had to say: "...(N)ot since Bhaskara (12th century) has she [India] produced a single native genius in

this field (of mathematics)"[16]. Similar sentiments were expressed by more recent historians of mathematics such as Boyer (1968, p. 244) and Eves (1983, p. 164).[17] The second reason is more important: and that is we believe the above paradigms to be subjective and/or unreliable.[18]

METHODOLOGY

In this project we use objective scientific paradigms, as stated explicitly in earlier publications[19], to investigate our conjecture. Specifically we would seek to find direct evidence of transmission and, in the event of the failure to discover direct evidence, also test the veracity of the extant belief that there was no transmission of the Kerala mathematics to Europe. Or to borrow an analogy from mathematical statistics, we aimed to test the *status quo* hypothesis (i.e., the null hypothesis) that there was no transmission of (influence of) the Kerala mathematics to (in) Europe against our conjectured hypothesis (i.e., the alternative hypothesis) that there was transmission and influence. The aim then was to study materials that were identified in the project proposals that could provide supporting evidence for either of the hypotheses. At the end of the study a judgement could be made on as to which of the two hypotheses could be supported by the evidence made available. The evidence would not be of the 'smoking gun' circumstantial type, but would be objective evidence of the process or act of transmission having occurred as identified in documents.

The manuscript and other materials that were identified in the project proposals for investigation together with their locations and the associated justifications for their inclusion are given below. Materials were initially identified by the project applicants and copies of those that appeared to warrant specialist translation and study were obtained and sent to our Research Assistants, Dr Carlos Goncalves and Dr Jean Michel Delire. Additionally Dr Carlos Goncalves spent an extended time in Portugal to map out the existence of relevant materials in the un-catalogued documents in the archives at the University of Coimbra; Dr Delire undertook a study of the catalogue of Oriental manuscripts in Leiden University.

A. The correspondence and reports of the Jesuit missionaries to the Kerala region in the period 1540 to 1650.

The arrival of Francis Xavier in Goa in 1540 heralded a continuous presence of the Jesuit missionaries in the Malabar (the region surrounding Kerala) till 1670. Some of the Jesuits arriving after 1578 appeared to have the objective of gathering information from India.[20] These Jesuits were trained either by Clavius or Grienberger (Clavius' successor as Mathematics Professor at the Collegio Romano) were sent to India. Most notable of these were Matteo Ricci, Johann Schreck and Antonio Rubino. Ricci had specialist knowledge of mathematics, cosmography, astronomy and navigation; the Jesuit historian Henri Bernard states that Ricci "...had been requested to apply himself to the scientific study of this new and imperfectly known country, in order to document his illustrious contemporary, Father Maffei, the 'Titus Livius' of Portuguese explorations."[21] Rubino had studied with the French mathematician Viete, well known for his work in algebra and geometry. At some point in their stay in India these Jesuits went to the Malabar region including the city of Cochin, the epicentre of developments in infinite senses.[22]

Indeed, a study prior to the present project identified the following items of circumstantial evidence to support transmission:

i) These Jesuit missionaries were interested in arithmetic, astronomy and timekeeping of the Malabar region.[23]
ii) These Jesuit missionaries able to appreciate this knowledge by their learning of the vernacular languages such as Malayalam and Tamil.[24]
iii) Local sciences such as astrology or *jyotisa* were included in the curriculum of the Jesuit colleges in the Malabar Coast.[25]
iv) There were descriptions of the sciences and the mechanical arts of the Malabar region sent to Rome.[26]
v) Ricci's enquiries in 1580 about Indian calendrical science.[27]

vi) Rubino's report in 1610 about the errors in European tables based on inferences from local calendrical knowledge.[28]

vii) The letter from Schreck, in 1618, of astronomical observations intended for the benefit of Kepler.[29]

viii) Rubino stated in a letter to Grienberger about the Malabar Brahmins who "he wrote, 'are devoted to study of the movements and aspects of the planets and stars, particularly of twenty seven by which they govern and rule.' He tried to learn their secret of predicting 'the hour and minute of eclipses of the sun and moon,' but was unsuccessful because they shared this knowledge only with relatives and in secret."[30]

Thus it was relevant to study the following mass of materials to determine whether or not there was <u>direct</u> evidence of transmission:

i) Archivum Romanicum Societate Iesu, ARSI, Rome. A study of mainly manuscript letters and reports from Jesuit missionaries to their headquarters in Rome.

ii) Gregorian University Archives, Rome. A study of manuscript correspondence of scientist Jesuits [amongst them Rubino, Ricci, Schreck] to Clavius and Grienberger.

iii) University of Coimbra archives, Coimbra. An investigation to identify any Jesuit correspondence and materials and to examine the works by the Jesuit mathematician Borri. He was the only Jesuit missionary who went to the Malabar and returned. He specialised in astrology.

iv) Ajuda library, Lisbon. A study of manuscript correspondence of the earlier Jesuit missionairies to the Malabar up to 1568.

B. The works of prominent Jesuit mathematicians in Rome who may have analysed the transmitted manuscripts of the Kerala mathematics

There appears to be a similarity in the approach to certain result in the proto-typical calculus in the *Yuktibhasa* and the approach

to calculus adopted by some Renaissance mathematicians. Additionally some results of Indian mathematics of 500 years earlier – those of Bhaskara II – were rediscovered in the Renaissance by Fermat and Wallis[31].

If the Jesuits were the conduit of the conjectured transmission then it would require Jesuit mathematicians of some ability to interpret the works of the Kerala mathematicians. The most prominent of the Jesuit mathematicians of the period in question were the two mentioned previously, Christopher Clavius[32] and Christopher Grienberger[33]. Additionally we know from our earlier studies[34] that scholarly Jesuit missionaries in the Malabar such as Ricci and Rubino were in correspondence with Clavius and Grienberger.

The materials researched were the:

i) Works and correspondence of Christopher Grienberger. The correspondence was identified from the edited correspondence of Clavius[35]. The mathematical works we was identified was his unstudied manuscripts GES 874 and GES600 in the Biblioteca Nazionale, Rome.

ii) Works and correspondence of Christopher Clavius. The correspondence was identified in the edited correspondence as mentioned above and the work identified was the one work which may have been susceptible to influence from Indian sources: *Theodosii Tripolitae Sphaericirum Libri III*, located in the Univeristaria Alessandrina in Rome. This deals with the calculation of the trigonometric ratios which we posited to have some connection with the Kerala mathematics.

C. Indian scientific MSS located in libraries in Europe

Our studies prior to the present project included an examination of the correspondence of Renaissance mathematicians organised by Marin Mersenne[36].The minim monk Marin Mersenne, through his correspondence with the leading scientists and

mathematicians of the early 17th century, was an important conduit for transmission of knowledge. In this correspondence there is mention of Brahmins[37] and of the orientalists Gaulmin[38], Erpen and his "les livres manuscrits Arabics, Syriaques, Persiens, Turcs, Indiens en langue Malaye"[39], Golius and Drusius in the University of Leyden[40].

Additionally our prior studies included an initial survey of catalogues in the Vatican library. These indicated the presence of a large number of palm leaf manuscripts in Malayalam (the language of Kerala) and Tamil (the language of the neighbouring state, Tamil Nadu). Thus we undertook an examination of manuscripts whose existence we discovered in our earlier studies. These consited of:

i) Malayalam and Tamil palm leaf manuscripts in the Vatican library.
ii) Oriental manuscripts in Leiden University.

RESULTS OF THE INVESTIGATION

The materials identified in the previously were studied in-depth. Being primary archival research the net that we cast over the materials was fine and therefore necessarily captured large amounts of information not directly relevant to the aims of the research. For example the further Jesuit correspondence examined at the ARSI revealed only material related matters related to the missions [conversions, finance, the establishment and administration of the colleges, etc]. Similarly the material at the Ajuda library did not reveal any evidence of scientific information gathering and reporting from the Jesuit missionaries in the Malabar. Examples of letters from the Ajuda are shown below:

> Folio 49-IV-50; Letter No. 192. Carta do Irmao Luis Frois para casas e collegios da companhia de Europa escrita em Goa ao derradeiro de nouembro de 1557 [fl 98] [- fl 108v: 20 pages. About Brahmans, moors, etc]

> Folio 49-IV-50; Letter No. 195. Copia de outro do Irmao Luis Frois do collegio de Goa a 14 de nouembro de 1559. [fl 120v] [-fl 131v: 22 pages. About Brahmans, moors, etc]

> Folio 49-IV-50; Letter No. 264. Parte dalguas cousas de o Irmao Luis Frois escreuo da India ao Irmao Volgango germano companhia de Jesus no collegio de Coimbra a 30 de nouembro de 1560. [fl 333v] [-fl 335v: 6 pages. About Brahmans, moors, etc]

We report that the evidence from the examination of the Jesuit documents and correspondence merely confirmed the findings of our prior study that there was strong motivation on the part of the Jesuit missionaries to acquire the science of the Malabar[41]. However, the evidence from the trawl of materials studied so far does not support the contention that the Jesuits acquired any of the manuscripts that contained the Kerala mathematics and neither was there any <u>direct</u> evidence that they acquired knowledge of these results from a third party.

The correspondence between Clavius and Grienberger on the one hand and the Renaissance mathematicians in contact with them [notably Adrian van Roomen] suggest that they were working in an epistemology which seemed uninfluenced by the Kerala mathematics. However, influences of the earlier Indian mathematics notably from that of Aryabhata were detected. In this context it should be pointed out that Clavius was essentially re-writing the earlier work on trigonometric tables by Regiomontanus and thus it will be truer to say that the latter rather than Clavius was influenced by Aryabhata. For in Regiomontanus's *Compostio Tabularum suinuum Rectorum*, published in 1541 he, in the manner of Ptolemy, calculates the sines of 90^0, 45^0, 60^0, 30^0, and 15^0 from appropriate triangles, and then uses the 'Pythagorean' theorem to calculate sine 75^0. He then uses the formula 'For a random arc in a quadrant, the sine is the middle proportional [geometric mean] between half the radius and the versed sine of double the arc' which is equivalent to Aryabhata's earlier (c 499 AD) *kramatkramajya* rule[42]. This finding is in accordance with Knobloch's work[43] on the Arab influence on Clavius's work as we note that Arab mathematics was itself influenced by developments in India[44].

Having said that, a more detailed comparative study needs to be made between the early European work on the construction of trigonometric tables and the Kerala work on similar tables. A discussion of the Kerala work is found in one of the papers in this volume entitled 'Kerala Mathematics: Motivation, Rationale and Method'.

Grienberger's hitherto unstudied manuscript works GES 674 and GES600 were examined in detail. It revealed a lengthy work on the computation of trigonometric tables graduated in degrees with an intended accuracy of 18 places of decimals. The novel methods and algorithms used in the lengthy calculations were deciphered with some difficulty with the assistance of the Research Assistants. It revealed that Grienberger's methods were outstanding and novel, utilising methods from many sources, but there was no evidence on influence from Kerala mathematics (although the arithmetic used is Indo-Arabic). We (and, implicitly, several others including C K Raju) had conjectured that the initial value of sine of 1 minute [correct to 22 places of decimals] used by Grienberger to generate his tables was calculated using the infinite series for sine that had been discovered earlier by the Kerala mathematicians. However our analyses made it clear that the infinite series of trigonometric ratios is impracticable for the construction of trigonometric tables graduated in degrees. Nevertheless the studies of Clavius and Grienberger, especially the latter, on the construction of trigonometric tables will be of interest to historians of mathematics and work is ongoing to publish these findings.

A thorough study of a set of uncatalogued Malayalam and Tamil palm leaf manuscripts at the Vatican library was undertaken by Dr Raghava Varier, a research scholar of Malayalam and Tamil literature. No evidence of Kerala mathematics or astronomy was found in these manuscripts. The manuscripts were mainly works on lexicography and catechisms with one or two works on medicine. A full report on the archival work undertaken by Dr Varier is contained in one of papers ('Uncatalogued Malayalam Manuscripts in Europe: A Report of Work carried out in Two Libraries In Rome') in this volume.

Amongst the oriental manuscripts at Leiden there were several that contained works of early Indian mathematics. For

example: Or. 2361. Sanskrit, paper, 42 ff. *Mahâsiddhânta* by Âryabhata II (fl. AD 950) in 18 *adhyâyas*. However the history of acquisition of these manuscripts was missing and thus nothing could be inferred about the possible transmission implications. Furthermore, the mathematical works related to a period of Indian mathematics prior to the emergence of Kerala School.

TENTATIVE CONCLUSIONS

The painstaking trawl of the mass of manuscript and other materials mentioned earlier in this Report has yielded no direct evidence of the conjectured transmission. We have to therefore report that on the basis of the evidence of the documents studied so far that the evidence supports the null hypothesis formulated earlier. Thus the European Renaissance developments of prototypical calculus may well have been independent of the developments in that subject in Kerala some centuries earlier.

This is only a provisional conclusion as it is by no means the case that all of the material that is required to be studied to reach a definitive conclusion has been studied. There may be materials available amongst the mass of un-catalogued documents in Portugal or in private libraries in Italy. Both these countries suffered upheavals in which library contents were dispersed – in Portugal the suppression of the Jesuits in 1750 by the Marquis of Pombal and in Italy the effects of the thirty years war in the early 17th century. Additionally, the unstudied works of other Renaissance mathematicians (such as Toricelli and Cavalieri) could be relevant to the examination of our hypothesis.

Also, the possibility of oral transmission does exist. However, weighed against this possibility is the fact, as far as can be determined, only Cristovão Borri of the scientifically accomplished Jesuit missionaries eventually returned from Malabar to Europe. And we have found no evidence of any information from him being available in his publications or reported on by the scholars with whom he was in contact. Nevertheless, it may be the case that the oral conduit may have been an Arab one with the sea route to Basra from Kerala being still active in the period in question. Given the new information,

some of it oral, on the crucial role of the Arabs in the creation of the Copernican Revolution, it may be useful to pursue this line of thought in any future research.[45]

In attempting to study transmissions, it is also important to examine in greater detail the context and the motives of the Jesuit missionaries who were sent to India and their mode of communication with one another. The primary motive was of course evangelical but to achieve this different strategies were adopted. In their mission to India and China, it was recognised early on that they sent missionaries who were both well informed on the sciences and technology of the day and could debate with local scientists that they came across. It is unlikely, given the nature of the relationship between themselves and the host societies (who were often described as pagans and unbelievers)[46], that they would openly concede or acknowledge the superiority of indigenous scientific knowledge. The position of the Jesuits was during the period under study quite precarious, as the Galileo Episode indicated. In any case, within Europe of that period, the intellectual climate did not favour any explicit acknowledgement of the debt owed to the work of others, unless they were ancient Greeks who were perceived as the fount of knowledge from which European history and thought emerged.

Now, from the beginning of the Jesuit missionary-scientific enterprise, great importance was placed on the Jesuits in far-flung places keeping in touch other. Francis Xavier, the first missionary in the East, urged the members to pay particular attention to the composition of the letters, both public and private, making the former suited to the needs. Ground rules were laid down on the manner in which communication was to be conducted.[47] It would seem unlikely that an organisation which had enemies in Rome and was perceived at times as sailing close to wind, particularly during the time of the Galileo Episode, would admit either in the letters or elsewhere that they were recipients of 'superior' knowledge in science from 'pagans and unbelievers'. So the absence of documentary evidence of transmission should not come as such a great surprise.

At any rate what we do know from the project study is that the null hypothesis of no direct transmission is sustained by the

evidence gathered. If in the final analysis – by extended archival work or from other sources over many years - this null hypothesis is confirmed then it will be the first major case of scientific development in the post Ancient era that has remained localised in its place of origin and that, despite the existence of a direct corridor of communication to Renaissance Europe.

What reasons might be put forward for this if in the final analysis the Kerala mathematics remained localised and un-influential despite the conditions for its ease of transmission being present? The combination of Jesuits colonial superiority and insensitive treatment of the Hindus by the Portuguese may have alienated the Brahmin keepers of the Kerala mathematics. The Brahmins were also possessive of this knowledge and only a privileged few were allowed access to its explications, Thus, as mentioned earlier, when Jesuit scientists like Rubino attempted to seek out this knowledge they were unsuccessful. Even if the Kerala mathematics reached Europe, attitudes of colonial supremacy in which Europeans saw themselves as 'civilizers' of the pagan colonies would have rejected the worth of the knowledge of 'barbarians'. That such an attitude existed is not difficult to accept given its later manifestations in the works of historians of science such as Sedillot[48]. If the conclusions of this research are confirmed in time *and* the reasons above are valid then this particular passage in the history of mathematics will be doubly noteworthy.

DISCUSSION

The discussion of this paper seemed to polarise between two opposing viewpoints. The more 'pessimistic' version believed that the barriers of transmissions were insurmountable, particularly when it came to communication in natural or mathematical language between the Jesuits and the native population. But this view would seem to underplay the importance given by the Jesuits to learning local languages to understand, influence and evangelise the local population Further, the point was made that the dearth of archival material, except for a document discussing Hindu mathematics in very general terms, should reinforce the 'pessimistic' viewpoint.

There were also criticisms of the undue weight given to the 'outdated' opinions of Van der Waerden and Neugebauer when it came to the issue of how to establish transmissions and of the neglect of the modern tendency to look 'outwards' in studying the history of mathematics in a cultural and thematic context. It was felt, however, that these criticisms ignored the influence of both Van der Waerden and Neugebauer on the history of mathematics, especially since their theories of transmission may be viewed as crystallisations of the perceptions that found widespread support among the conservative Eurocentric intelligentsia.

The core criticism of the "optimistic" wing was that the paper was unduly hasty in accepting the 'no-transmission' thesis. This arose from the somewhat arbitrary decision to reject the criteria used to establish transmissions by Neugebauer (the identification of methodological similarities, the existence of communication routes and a suitable chronology for the transmission) and van der Waeden (in the history of science independent inventions are exceptions, and the general rule is dependence) by arguing in the paper that such criteria are "subjective and/or unreliable". Instead, the paper set out to assume the hypothesis of no-transmission as the default (or 'null') hypothesis, unless *direct* evidence for transmission existed. . On this basis, the paper concludes at least provisionally that the European Renaissance developments of prototypical calculus may have been independently discovered.

But this ignores the extensive *circumstantial* evidence supporting transmission, including the interest of the Jesuit missionaries in arithmetic, astronomy and timekeeping of the Malabar region, and their capacity to appreciate this knowledge in the vernacular languages by virtue of their training; the inclusion of the local sciences such as astrology in the curriculum of the Jesuit colleges in the Malabar Coast; and the similarities in the techniques adopted by European mathematicians (e.g. Fermat and Wallis) to those of their Indian predecessors. Also, there may be other reasons for the absence of evidence. In particular it is possible that Jesuits may have been unwilling to openly acknowledge being recipients of 'superior' knowledge from 'pagans and unbelievers'.

In any case, it was argued by some, the absence of evidence for transmission is *not* evidence for no-transmission. It only becomes so because of the adoption of the restrictive no-transmission theory as the Null hypothesis. Otherwise, the balance of evidence presented would suggest not only that there exists strong circumstantial evidence for transmission. And the Jesuits need not have been the only conduit for transmission of Kerala mathematics to Europe.

ENDNOTES

1 See, for example, M Baron, *The Origins of the Infinitesimal Calculus*, Oxford, Pergamom, 1969, p 65.

2 See, for example, C Edwards, *The Historical Development of the Calculus*, New York, Springer-Verlag, 1979, p189, and V Katz, "Ideas of calculus in Islam and India", *Mathematics Magazine*, Washington, 68 (1995), 3: 163-174, p 163 and p 164.

3 See the work of K V Sarma, *A History of the Kerala School of Hindu Astronomy*, Hoshiarpur, Vishveshvaranand Vedic Research Institute, 1972, p 21 and p 22 and the paper by C M Whish, "On the Hindu quadrature of the circle and the infinite series of the proportion of the circumference to the diameter exhibited in the four Shastras, the Tantrasamgraham, Yukti-Bhasa, Carana Padhati, and Sadratnamala", *Transactions of the Royal Asiatic Society of Great Britain and Ireland*, London, 3 (1835): 509-523, p 522 and p 523.

4 For example, C M Whish, "On the Hindu quadrature of the circle and the infinite series of the proportion of the circumference to the diameter exhibited in the four Shastras, the Tantrasamgraham, Yukti-Bhasa, Carana Padhati, and Sadratnamala", *Transactions*, 3 (1835): 509-523; C T Rajagopal and M S Rangachari, "On an Untapped Source of Medieval Keralese Mathematics", *Archive for History of Exact Sciences*, Baltimore, 18 (1978): 89-102; C T Rajagopal and T V Vedamurthi, "On the Hindu proof of Gregory's series", *Scripta Mathematica*, New York, 18(1951): 91-99

5 For example, M Baron, *Origins of Calculus*, op cit, p 62 and p 63; R Calinger, *A Contextual History of Mathematics to Euler*, New Jersey, Prentice Hall, 1999, p 284. V Katz, "Ideas of calculus in Islam and India", *Mathematics Magazine*, Washington, 68 (1995), 3: 163-174, p 163 and p 164.

[6] R Rashed *Indian Mathematics in Arabic*, in Ch. Sasaki, J.W. Dauben, M. Sugiura (éds), *The Intersection of History and Mathematics*, Basel, Boston, Berlin : Birkhaüser Verlag, 1994, p. 143-148. John Berggren, *Episodes in the Mathematics of Medieval Islam*, New York, Springer Verlag, 1986, p 30

[7] S Benedict, *A Comparative Study of the Early Treatises Introducing into Europe the Hindu Art of Reckoning*, Ph.D. Thesis, University of Michigan, Rumford Press, 1914

[8] B V Subbarayappa and K Venkateswara Sarma, *Indian Astronomy-A Source-Book*, Bombay, Nehru Centre Publications, 1985, p XXXVIII

[9] B V Subbarayappa and K Venkateswara Sarma, *Indian Astronomy*, op cit, 1985, p XXXVIII

[10] O Neugebauer, *The Exact Sciences in Antiquity*, New York, Harper, 1962, p 166 and p 167

[11] B van der Waerden, *Geometry And Algebra In Ancient Civilizations*, Berlin, Springer-Verlag, 1983, p 211

[12] B van der Waerden, *Geometry And Algebra*, op cit, p 133 .

[13] B van der Waerden, "Pells equation in Greek and Hindu mathematics", *Russ Math Surveys*, 31 (1976): 210-225, p 221

[14] B van der Waerden, "Pells equation in Greek and Hindu mathematics", *Russ Math Surveys*, 31 (1976): 210-225, p 221

[15] See, D Almeida and G G Joseph, "Eurocentrism in the history of mathematics: the case of the Kerala school", *Race and Class* 45(4), 2004, 45-60

[16] D E Smith, *History of Mathematics*, Dover, New York, 2 vols, vol 1, p435

[17] C B Boyer (1968) *A History of Mathematics*, John Wiley, New York; H Eves (1983) *An Introduction to the History of Mathematics*, 5th Edition, Saunders, Philadelphia

[18] In the face of these unrigorous and non-uniform paradigms, we have proposed elsewhere (See The Aryabhata Group, 2002, 'Transmission of the Calculus from Kerala to Europe' in the *Proceedings of the International Seminar and Colloquium on 1500 Years Of Aryabhateeyam*, Kerala Sastra Sahitya Parishad, Kochi, 33-48) to adopt a standard of evidence sufficient for legal purposes. Briefly, we propose to test the hypothesis of transmission of the calculus from India to Europe on the grounds of (1) motivation, (2) opportunity, (3) circumstantial evidence, and (4) documentary evidence. However, in this paper, we focus our attention on the availability of documentary evidence.

[19] See D Almeida and G G Joseph, "Eurocentrism in the history of mathematics...", *ibid* and the Aryabhata Group, 2002, 'Transmission of the Calculus from Kerala to Europe' , *ibid*

[20] The Jesuits are generally perceived as the mediators of Western Science, especially in China. What is indicated here is another role: as intelligence-gatherers to fill some critical gaps that had appeared in Renaissance Europe, namely navigation and calendar construction.(See The Aryabhata Group, 2002, *Ibid*, pp. 35-38 for more details). They will not be examined here

[21] H. Bernard H, *Matteo Ricci's Scientific Contribution to China*, Hyperion Press, Westport, Conn., 1973, p 38

[22] See, for example, I. Iannaccone, *Johann Schreck Terrentius*, Instituto Universitario Orientale, Napoli, 1998 and U. Baldini, *Studi su filosofia e scienza dei gesuiti in Italia 1540 - 1632*, Bulzoni Editore, 1992. Ugo Baldini states ""It can be recalled that many the best Jesuit students of Clavius and Geienberger (beginning with Ricci and continuing with Spinola, Aleni, Rubino, Ursis, Schreck, and Rho) became missionaries in Oriental Indies. This made them protagonists of an interchange between the European tradition and those Indian and Chinese, particularly in mathematics and astronomy, which was a phenomenon of great historical meaning" (Baldini, 1992, p 70. My translation)

[23] See, for example, Josef Wicki, *Documenta Indica*, 16 volumes, Rome, Monumenta Historica Societate Iesu, 1948-, vol IV p 293, vol VIII p 458

[24] *Documenta Indica,* vol XIV p 425 and vol XV p 34*

[25] *Documenta Indica*, Vol III, p 307

[26] In the folio Goa 58, Jesuit library, ARSI, Rome

[27] *Documenta Indica*, vol XII, p 474

[28] U Baldini, *Studi su filosofia* ... 1992, p214.

[29] Iannacone, *Johann Schreck Terrentius*, 1998, p58

[30] P. D'Elia, *Galileo In China: Relations through the Roman College Between Galileo and the Jesuit Scientist-Missionaries* (1610-1640) Translated by Suter R and Sciascia M, Harvard University Press, 1960, p15

[31] For details see, The Aryabhata Group, 2002, 'Transmission of the Calculus from Kerala to Europe' in the *Proceedings of the International Seminar and Colloquium on 1500 Years Of Aryabhateeyam*, Kerala Sastra Sahitya Parishad, Kochi, 33-48

[32] The position of Clavius in the history of science is assured by his role in the Gregorian calendar reform and his many publications.

Clavius also maintained contacts with scientists and mathematicians outside Rome. Furthermore Clavius was influenced by Arab mathematicians – see E Knobloch *Christoph Clavius (1538-1612) And His Knowledge Of Arabic Sources*, http://www.ethnomath.org/resources/knobloch.pdf

33 According to M Gorman, Grienberger was "...a revisor of mathematical works written by Jesuits and in his strategies of engagement in epistolary relationships with natural philosophers and mathematicians outside the Jesuit order." (M Gorman, "Mathematics And Modesty In The Society Of JesusThe Problems Of Christoph Grienberger (1564-1636)", *The New Science and Jesuit Science: Seventeenth Century Perspectives*, ed. Mordechai Feingold, Dordrecht: Kluwer, 2003 (Archimedes vol. 6), pp. 1-120

34 See The Aryabhata Group, 2002, 'Transmission of the Calculus from Kerala to Europe' in the *Proceedings of the International Seminar and Colloquium on 1500 Years Of Aryabhateeyam*, Kerala Sastra Sahitya Parishad, Kochi, 33-48

35 U Baldini and P. D. Napolitani Corrispondenza / *Christoph Clavius ; edizione critica a cura*: Università di Pisa, Dipartimento di matematica, 1992

36 *Correspondance du P. Marin Mersenne*, 18 volumes, Presses Universitaires de France, Paris, 1945-.

37 *Correspondance du P. Marin Mersenne* Vol XIII, Page 518-521; a letter from Mersenne to Buxtorf. Mersenne mentions the knowledge of Brahmins and 'Indicos'

38 *Correspondance du P. Marin Mersenne* Vol XIII, Page 518-521

39 *Correspondance du P. Marin Mersenn.* Vol II, Pages 103-115

40 *Correspondance du P. Marin Mersenne* Vol II. Page 155; a letter from Mersenne to Rivet.

41 For details see, D Almeida and G G Joseph, "Eurocentrism in the history of mathematics: the case of the Kerala school", *Race and Class* 45(4), 2004, 45-60

42 K. S. Shukla and K. V. Sarma, *Aryabhatiya of Aryabhata*, New Delhi, Indian National Science Academy, 1976

43 See E Knobloch *Christoph Clavius (1538-1612) And His Knowledge Of Arabic Sources*, http://www.ethnomath.org/resources/knobloch.pdf

44 See, for example, J Berggren, *Episodes in the Mathematics of Medieval Islam*, New York, Springer Verlag, 1986 and R Rashed *Indian Mathematics in Arabic*, in Ch. Sasaki, J.W. Dauben,

M. Sugiura (éds), *The Intersection of History and Mathematics*, Basel, Boston, Berlin : Birkhaüser Verlag, 1994, p. 143-148.

45 Dr Balasubramaniam's paper in this volume could provide a useful pointer to the need to look beyond documentary evidence in establishing transmissions. See also Arun Bala *The Dialogue of Civilizations in the Birth of Modern Science*, New York, Palgrave Macmillan, 2006

46 From the very beginning, the founder of the Jesuit Order, Ignatius Loyala, saw that the missionary labours of the Jesuits should be primarily directed among the 'pagans' of India, Japan, China, Canada, Central and South America. Only with time did their attention turn to other Christian countries. As the object of the society was the propagation and strengthening of the Catholic faith everywhere, the Jesuits naturally endeavoured to counteract the spread of Protestantism. They became the main instruments of the Counter-Reformation; the re-conquest of southern and western Germany and Austria for the Church, and the preservation of the Catholic faith in France and other countries were due chiefly to their exertions

47 Polanco, an assistant to Ignatius laid down that three things had to be considered regarding any letters sent to Rome. First, what was to be written; next how it was to be written; third, with what diligence it was to be written and despatched. Polanco also gave detailed directions about the distribution of the letter and who had access to them. For further details see J Correia-Alfonso (1969) *Jesuit Letters and Indian History 1542-1773,* Oxford University Press, Bombay, Chapters 2-4

48 See, D Almeida and G G Joseph, "Eurocentrism in the history of mathematics: the case of the Kerala school", *Race and Class* 45(4), 2004, 45-60

12

THE JESUIT MATHEMATICIANS IN INDIA (1578-1650) AS POSSIBLE INTERMEDIARIES BETWEEN EUROPEAN AND INDIAN MATHEMATICAL TRADITIONS

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1. The Studies On Jesuit Scientific Activities In India (Mid XVI – Mid XVII Centuries)

Anyone considering the studies on the Jesuits' scientific role in Asia before their suppression in 1773, and specially during the first century of their history, is struck by great differences in the number of books or articles devoted to different regions. In particular, when the three more relevant cases are considered, it's immediately evident that most analyses concern China, while almost none deals with India and Japan.

Japan's case – even if much remains to be examined - is easily explained by the shortness of the time (ca. 1570-1610) during which the Society of Jesus was allowed to establish itself before being persecuted and expelled by the Tokugawas. In those years it advanced much on the religious ground, gaining a surprising number of conversions, but it could not develop a system of schools – two or three excepted - nor go very deep into

the country's learning (not to say scientific traditions) and starting a scientific interchange.[1 2]

India's case is, however, very different and – at a first sight at least - much less explicable.[3] The Jesuits established themselves in the subcontinent earlier than in China, and remained until their suppression; moreover, in the first century at least, more of them worked in India than anywhere else in Asia, and their establishments in the subcontinent were more numerous and widely distributed. This applies specially to the setting up of a school system, which one could take as a precondition for cultural interchange and scientific research. Immediately after 1550, Goa's college was raised to an university level, offering a curriculum ranging from *grammatica* (a three years study of Latin grammar and syntax) to *rhetorica* (a two years study of Greek and Roman classics), *philosophia* (usually a three years course in logic, natural philosophy and metaphysics, to which sometimes ethics was added) and *theologica* (usually a four years course in scholastic theology, moral theology, Holy Writings, to which sometimes Hebrew, canon law and "controversies", i.e. anti-protestant polemic, were added). About 1580 Cochin's college began to offer the same curriculum, and in the following years a number of Jesuit schools along both the Goa-Malabar and the Coromandel coasts reached the *grammatica* or *rhetorica* level. On the contrary, until late XVIIth century the Society was not allowed to establish permanent schools on China's mainland; as for Macao's college, formally on the same level as Goa's, it must be remembered that it belonged to the Japanese – not to the Chinese – province, and that it could never offer such continuous and well articulated courses as Goa (and Cochin) could.[4]

Also in general, the number of contributions on the Jesuit presence in China until mid XVIIth century exceeds that regarding India, and the amount of studies on different missionary areas in Asia does not conform to a dimensional scale of the Society's activities in those areas. However, as for scientific history the difference is much greater;[5] and there are some special reasons are to be found for it.[6]

2. Reasons for the Difference

In fact, the disproportion is not a mystery. If it does not reflect a quantitative difference in the Society's activities and establishments, it reflects an important qualitative one. As it has been documented many times, the Jesuit superiors in the Asian missionary provinces, in the Portuguese headquarters and in Rome were informed (mainly, but not only, by Ricci's reports) that "mathematics" (in a very broad sense of it) was perhaps the most appropriate key to open the locked doors of the Middle Empire. So, among the not many mathematicians the Society was able to produce in its first decades, practically all were destined to be sent to missions in China, virtually excluding both India and the American colonies of Spain and Portugal.[7] So a basic question arises: why did the Jesuits, until mid XVIIth century at least, consider mathematics as an essential missionary tool for China and not for India? And, consequently: did this judgement reflect some real differences between the two contexts, or did it mainly reflect the fact that in one of them there was a man (Matteo Ricci) whose depth of insight was unparalleled by anyone in the other?[8]

Adequate answers to both questions would require detailed analysis; in fact, no real attempt to answer them has been produced so far, and few of the data necessary for such answers have been collected. Some essential circumstances, however, may be considered as established:

(i) While the Jesuits in China produced both many translations and vulgarizations of European scientific (mainly mathematical) works and descriptions/ discussions of the country's philosophical and scientific doctrines (partial or incorrect as they may appear today), those in India produced nothing of the first genre and very little of the second (concerning only the theological-metaphysical ideas);

(ii) The little Jesuit scientific production in India known so far (some astronomical observations, mainly of eclipses and comets; some cartographical work) did not require advanced knowledge and methods, was occasional in its origin and was not the outcome of a specially devoted

group of specialists, as it is true for that in Beijing's mission since 1601;

(iii) Until the beginning of the XVIIIth century, the Jesuits' judgements on Chinese and Indian scientific traditions known so far show a more positive appreciation of the first than of the second. Given that this attitude lasted for more than one century, the fact that the Society did not consider mathematics as an useful missionary device in India cannot be referred to the absence of an Indian equivalent of Ricci, but only to some deep differences (true or false) perceived as existing between China's and India's socio-cultural reality.

Thus, it is only natural that little effort has been made to discover and examine the documents of what did not seem to deserve a greater one. Moreover, until very recently the few historical studies concentrated on the scientific research (mostly in geography and astronomy) performed by the Indian Jesuit missionaries in behalf of the Society's aims or of those of European science.[9] This means that such a research – episodes as the scientific conversations with Akbar, king Venkata II of Vijayanagara or some Brahmins with some Jesuits apart - has been considered as produced by an European colony overseas, little affected by the local context: that is, as a detached part of the European scientific enterprise. This accounts for another conspicuous difference between the historiography on the Chinese and the Indian missions, because in China's case the scientific work performed for the sake of Europe has been paid no more attention (indeed, probably a lesser one) than that devoted to impress a local audience: that is, the missionaries' scientific intermediacy.

3. Intermediacy

Intermediacy is inherently bidirectional. Sometimes only one direction seems real (or relevant), but the very fact of a transmission affects both poles of the process: Europe's ships reached the East while no Asian ship reached the West, and the Westerners did not introduce something definitely Asian in the

design and furniture of their ships after seeing the Asian ones; but it cannot be doubted (in fact, it is proved) that the experience of different forms, solutions and materials favoured new conceptions in European naval constructions. This notwithstanding, even in China's case an east-west direction in the Jesuits' scientific intermediacy has been considered in a rather limited sense. While their role in carrying information eastwards has been studied as substantial in promoting China's knowledge of more powerful methods and concepts (that is, in making it familiar with "modern" science), what they carried westwards has been appreciated as merely cultural (that is, as something precious for historical study much more than for scientific progress). To say it differently, the possibility that China's scientific conceptions transmitted to Europe by the Jesuits could affect in some relevant sense the development of western science has never been denied explicitly, but simply because they were judged so archaic to make such a possibility unreal.[10]

However, in India's case this could be false. As western scholarship discovered only around 1840, until middle XVIth century some parts of Indian science – specially the kind of computational mathematics developed in the south-western part of the country - were not only on a same level, or even more advanced than contemporary European science, as is the case for some Chinese results and conceptions (not in pure mathematics, however). They preceded it by centuries and some results and conceptions, although embodied in conceptual patterns very different from the European ones, were equivalent to some attained in Europe only from mid seventeenth-century: that is, by the mathematics of the "scientific revolution".[11] In principle, it has been conjectured, those results and conceptions were such that their transmission to Europe could have been essential in promoting the transition of European mathematics from classical geometric models to the computational: that is, one of the basic events in the history of western science.[12] So, paradoxically, a westwards Jesuit intermediacy could have been specially important just in that case for which it has been long excluded.

It is a truism to say that a supposed historical fact may be ranged among real events only if it satisfies two conditions, the

first logical and the second factual: it must conform to some basic requirements (i.e., it cannot hypothesize counter-to-fact circumstances); it must be supported by documentary evidence. And, obviously, the first condition makes the second possible, but only the latter transforms possibility into reality. In concrete terms, this means that to prove the westwards transmission hypothesis one has:

1) to suppose – or document - that one or more Jesuits could get (or, in fact, got) an inner knowledge of some of the more difficult parts of Kerala's mathematical works, little known even among the local students;
2) to document (or to suppose) that this knowledge was transmitted to Europe;
3) to document (not to suppose) that it reached some of the founders of the new mathematics. Obviously again, to document (3) makes (1) and (2) real, even if no proof of them is found; to document (2) makes (1) real, even if no proof of it is found, but it makes (3) only more plausible; to document (1) only makes (2) and (3) more plausible. So only a documentary proof for (3) may transform the transmission hypothesis into reality. The present state of documentary research regarding (2) and (3) is the concern of some other papers in this symposium; so this one focuses on two different points: the general possibility for some Jesuit to become acquainted (and to understand fully) some advanced Indian mathematical text or doctrine; the kinds of Jesuit texts which could be the vehicle of a transmission, and where to concentrate the search for them.

Before discussing these two points, however, it is important to circumscribe the historical period in which the transmission could happen.

4. The Chronological Terms Of A Possible East-West Transmission

In a general sense the Jesuit intermediacy between Europe and Asia, in both directions, lasted from the 1540s until the Society's

provinces in Asia were suppressed (1759-1768) and even after, because many missionaries remained in the East until their death.[13] However, the relevant period for the transmission hypothesis is much shorter. A lower limit may be put in 1578, when Matteo Ricci was the first Jesuit missionary endowed with some mathematical learning to reach India. The upper one cannot exceed 1640 or little more, because the transmission must have occurred before the mathematical advances in Europe it could have favoured. So any research may be focused on missionaries in India during about 60 years (that is, on 100-200 persons), and on Jesuit letters and reports written in the same period.[14] This is important not only because it circumscribes the number of persons and circumstances to consider, but also because it circumscribes the provenances of the relevant persons, and consequently the forms of their mathematical training and the places to which they could send their letters and reports. In fact, as shown in the following paragraph, only a very little fraction of the Jesuit missionaries could manage sophisticated mathematics (specially if they met with "alien" forms of it), and this fraction was represented by those who received a (relatively) advanced mathematical training before leaving Europe. Now, until 1640 most Jesuit missionaries overseas endowed with some mathematical competence came from the mathematics school of the Collegio Romano, the so called Academy of mathematics; in fact, the school's primacy on those in the other great colleges of Europe ceased about that same year, so that the possible period for a transmission roughly coincides with that of that primacy.[15] As a consequence, most of the potential agents of the transmission were Italians or former students in Rome.

5. Circumscribing the Possible Individual Agents of the Transmission

The transmission hypothesis supposes that one or more Jesuits came in touch with Kerala's advanced mathematics, understood it and perceived its value in front of Europe's mathematical tradition. It is pertinent to ask which qualifications should be involved in such an event. Theoretically, the more obvious scenario would be: someone gained access to the ancient, reserved

texts (perhaps through a Brahmin acquaintance); this person could read their Sanskrit and was such a mathematician to extract the substance out of the involved, poetical/allegorical and obscure (for an European) modes of expression and non-canonical (for an European) demonstration procedures; he was also able to appreciate its value in front of the "state of the art" in Europe. Below this ideal situation, some lower degrees may be imagined: that person could have the texts translated by another (European or Indian); he could not have read the texts (original or translated), but only have been informed on certain conceptions, methods and results during conversations with local students.

As evident, the ideal situation implies a number of circumstances which could hardly occur together: a Hindu - presumably a Brahmin - should have allowed a foreign person, who was also his rival in religion, to get in touch with precious and reserved texts; the other person should have been highly proficient in both Sanskrit and mathematics; he should also have been so patient and curious about those texts' content to overcome the barrier produced by unusual expressions and demonstration procedures. In front of this, it must be recalled that, as a matter of fact, no hint about Indian mathematical texts – perhaps some elementary ones excepted - is found in Jesuit sources before 1650 at least, and no such qualified person seems to have existed among India missionaries in those years; in particular, only De Nobili and, perhaps, Fenicio (both far from being mathematicians) seem to have been proficient in Sanskrit, while Rubino – for instance - only mastered Telugu and Malayalam.[16]

Moreover, even leaving the linguistic factor aside, the mathematical aspect itself raises some problems, both in the ideal case and in that of transmission through conversations. The programme of mathematics adopted in the Society's schools since about 1550, made official in the *Ratio atque institutio studiorum* (1599), only included elementary geometry and arithmetic, the 'sphere' (elementary spherical astronomy), some notions of geography, calendar theory and (sometimes) gnomonics.[17] This excluded advanced arithmetic (in the Diophantean or any other sense) and algebra, advanced (Archimedean and Apollonian) geometry, trigonometry, the mathematical theory of planetary orbs, statics and hydrostatics:

briefly, all advanced sectors of the mathematical sciences at that time. Therefore, no person formed only on that programme could be able to appreciate – let alone to understand fully – such refined mathematics like that of the Kerala's school. In addition, even the ordinary mathematics course was not taught in most Portuguese (and Spanish) Jesuit schools until after middle seventeenth century; this means that most missionaries had not attended it during their studies in Europe, and so lacked even the basic notions mentioned before.[18]

From this two consequences follow, both important for the transmission hypothesis. The first is that – except when some special extra-curricular studies are documented - Jesuits who were formed only in the Society's canonical curriculum (that is, decidedly more than 90% of the missionaries until after 1650) may be excluded as possible agents of the transmission. The second is that, if an agent (or perhaps agents) existed, most probably he is to be found among the non Iberian missionaries. In fact, all the Jesuits in India and the Far East for whom a more than elementary mathematical competence is documented were not Iberians; with one or two exceptions, they had acquired that competence attending advanced courses in the "mathematics academies" existing in very few colleges, the Collegio Romano apart.[19] This is documented not only by the national origins and study places of those who showed some technical knowledge in pure mathematics, physico-mathematics and astronomy, but by a great number of letters of Jesuit superiors complaining about the Iberian provinces' inability to train mathematicians and bitterly reproaching the local superiors about it.[20] A mere list of mathematicians sent to Asian missions before 1650 sufficiently represents the situation:

TABLE I

Presence in India	*Name*	*Nation*	*Final Destionation*[21]
1578-79 (or 1580)	Michele Ruggieri	Italian	China
1578-82	Matteo Ricci*	Ita.	China
1588-89	Michele Ruggieri[22]		
1597-1601/2	Muzio Rocchi*	Ita.	Japan, China

1599-1601	Carlo Spinola*	Ita.	Japan
1601-1603/5	Manuel Dias	Portuguese	China
1602-38	Giovanni Antonio	Ita.	India (then Japan)
1602-03	Rubino#	Sabatino De Ursis*	
	Ita.	China	
1608-10		Giulio Aleni#	
	Ita.	China	
		Francesco Sambiasi	
	Ita.	China	
1615-16	Cristoforo Borri#	Ita.	Vietnam
1615-18	Jan Wremann*	Croatian	China
1618-19	Johann Schreck (Terrentius)#*	German	China
	Johann Adam Schall von Bell*	Ger.	China
	Wenceslaus Pantaleon Kirwitzer#	Bohemien	China
1618-22	Giacomo Rho#*	Ita.	China
1623-24	Cristoforo Borri[23]	Ita.	
1629-43	Johann Chrysostomus Gall#	Ger.	India
1640-42	Martino Martini*		
1640-43	Ita.	India	
1643-44	Antonio Maria Costantini#	Ita.	China
1647-67	Hendrick Uwens (Buys, Busaeus)#	Belgian	India[24]

Nineteen persons in about 70 years are not an imposing number, but as regards a possible agent of the transmission this number is still too large. It results from considering not only the missionaries who remained in India until their death or for a long time, but also those for whom Malabar was only a halting place in their travel to eastern Asia or east Africa (owing to the winds governing navigation from the Cape of Good Hope to Nagasaki, Jesuits reaching Goa or Cochin -usually between mid-September and mid-October- but destined to the Far East or to East Africa waited there from five to ten months for ships sailing to Malacca-Macao-Nagasaki or to the African coast from Kenya to

Somalia).[25] During those months many missionaries pursued their study of theology and also began studying the local languages, but they could also have occasions to meet with representatives of the local culture; in principle, it cannot be excluded that some of them could establish some connections with cultivated Indians and even gain access to Sanskrit texts. This applies more to those who – for causes ranging from health reasons to the completion of the formal course of theology to still others – remained in India for 3 or 4 years before sailing to their final destination – (Ricci, Rocchi, Wremann, Rho). It is documented that sometimes in such intervals scientific research was made, and it seems that Ricci at least tried to get information about the local calendar.[26] However, in front of Rubino's declared failure during a seven years stay in India (1602-1609), the possibility that in the space of some months, or even a few years, someone managed to have access to the most specialized parts of local mathematics and could study them appears theoretical much more than effective.[27] This rather negative evaluation is confirmed by a cursory analysis of the subgroup that must be considered first: those persons in the list (the same Rubino, Gall and Uwens) for whom India was a stable destination.

Rubino's extant letters or writings and information on him by others have been collected and examined, because his tragic death in Japan (1643) attracted attention on him not as a mathematician or a Jesuit superior, but as one of the Christian martyrs in the Far East.[28] Here it is sufficient to observe that, in a thirty years' sojourn in south India, he never mentioned to have overcome the denials he had met with in his first attempts.

Although the least known of the three, Gall looks a promising figure. An accomplished mathematician and astronomer (he had been a pupil of the most renowned mathematical school in the Jesuits' German colleges, that of Ingolstadt, and before sailing to India he had taught mathematics in Lisbon from 1620 to 1626 or 1627), he was in different residences around Goa until his death in 1643. Unfortunately, no letters or writings by him during his Indian years are known, and information in the Society's catalogues or other sources are scant.[29]

Uwens, best known for his staying at the Moghul court as a teacher of one of Aurangzeb's sons, had also taught mathematics in Lisbon, from 1642 to 1645 or 1646. The emperor's support and his deep experience of the country could make him a likely candidate, but he reached India toward the end of the period which is relevant here; moreover, he does not seem to have sojourned (longly at least) in Kerala and no mathematical letters or writings sent to Europe by him are known.[30]

Thus, unless new evidence is found and some basically new circumstance is established, the only possible deduction seems to be that not only no information exists on a Jesuit mathematician having managed to study some advanced Indian text (not to say to transmit it, or its content, to Europe), but no serious clue appears of a scientific interchange not purely superficial and more than occasional.

6. Scientific Teaching and Research in the Indian Missions of the Jesuits

It could be hypothesized that, during the navigation to India or while in Goa or Cochin, some of the specialists taught some advanced mathematics courses to their fellow brothers, so that some of the latter could become proficient enough to cope with algebra, progressions and series. There is proof that such courses were taught *occasionally* during navigation, in the Indian colleges and in Macao.[31] However, the information found on those courses usually mentions basic astronomy and pure and applied geometry, not advanced computation. Moreover, if systematic and extended, that kind of training would result in producing a group of local specialists, sufficient to meet the missions' technical and architectural needs and to enable Malabar's major schools, at least, to offer courses on mathematics within the philosophy course, as required by the *Ratio studiorum*. On the contrary, this happened most rarely, as a list of courses in Goa's and Cochin's colleges shows:

TABLE II

The "upper" disciplines taught in Malabar's Jesuit colleges (1555-1650)[32]

1555	Goa:	logic
1561	Goa:	theology (theol), philosophy (phil)
1562(?)	Goa:	theol., logic
1564	Goa:	theol., logic
1565(?)	Goa:	theol., phil.
1567-68	Goa:	logic
1568	Goa:	theol., phil. (8 students)
1569	Goa:	phil.
1571	Goa:	theol., logic (23 students)
1572	Goa:	phil. (8 students)
1573	Goa:	phil. ("many" students)
1575	Goa:	theol., logic
1576	Goa:	theol. [no philosophy]
1577	Goa:	logic
1578	Goa:	phil. (19 Jesuit students, plus others)
1579	Goa:	phil.
1580	Goa:	theol. [no philosophy]
1581	Goa:	logic (16 students)
1582	Goa:	phil.
1583-86	Goa:	[no philosophy]
1589	Goa:	phil.
1587	Goa:	theol., phil.
1593-94	Goa:	[no philosophy]
1594	Goa:	theol., phil. (15/18 students: 3 were born in India)
1595	Goa:	logic (11 students)
1596	Goa:	theol., phil.
1597	Goa:	theol.
1599	Goa:	theol., phil.
1603	Cochin:	theol.
1601/3	Goa:	theol., phil.
1603/4	Goa or Cochin:	mathematics (G.A. Rubino)
1605	Cochin:	theol., phil.
1606	Goa:	theol., phil.
	Cochin:	theol., phil.
1608	Goa:	theol., phil.
	Cochin:	theol., phil.

1609	Goa:	theol., phil.
	Cochin:	theol., phil.
1610	Goa:	theol., phil.
	Cochin:	theol., phil.
1611	Goa:	theol., phil.
1612	Goa:	theol., phil.
1613	Goa:	theol., phil.
1614	Goa:	theol., phil.
1615	Goa:	theol., phil.
1618	Goa:	theol., phil.
1619	Goa:	theol., phil.
1620	Goa:	theol., phil.
1621	Goa:	theol., phil.
1623	Cochin:	theol., phil.
1624	Goa:	theol., phil.
1627	Cochin:	theol., phil.
1628	Cochin:	theol., phil.
1634	Cochin:	theol., phil.
1641	Goa:	theol., phil.
1647	Goa:	theol., phil.
1648	Goa:	theol., phil.
1649	Goa:	theol., phil.

The absence of a teaching of mathematics apart, everything suggests that no appreciable knowledge of mathematics existed except in the men already mentioned. Thus, as for possibility of discovery and transmission, it may be said safely that it cannot be excluded in principle, but it cannot be accounted for on the basis of the didactical and cultural standards of the Jesuit missions in Kerala (and, generally, in whole Malabar) and the known qualifications of the missionaries there. Now, passing from possibility to reality, what can be said about the potential channels for a transmission and the European terminals of it?

7. The search for Documentary Evidence

The Society's written communications to and from Asia may be considered from two viewpoints: the travel routes; the different kinds of documents into which they may be classified. From the first point of view, a difference exists between the Indian missions and those in the Far East. As for the latter, until about

1610 messages to or from missions pertaining to the Portuguese Assistance (China, Japan, Vietnam) travelled on the Portuguese "Carreira da India": Lisbon-Cape of Good Hope-Goa-Malacca-Macao-Nagasaki (or the reverse); messages to or from those of the Spanish Assistence (Philippines) travelled first on the Atlantic route (Sevilla-Cuba-Veracruz), then on that of the "Manila galleons" (Acapulco to Manila), or the reverse. After 1610, the Dutch conquest of Malacca endangered the Eastern part of the Carreira; thus, to assure the arrival of a message, more copies of it could be sent along different routes at a time (the Carreira, the Spanish route Manila-Acapulco, the Dutch route from Batavia and others also). Communications to and from India, on the contrary, went on in the traditional way until the Society's suppression. For this reason, while a message coming from east-than-Malacca Asia after 1610 may survive in more copies (possibly conserved in different places), those sent from there before 1610, and all those from the Indian and African regions, were usually written in just one copy so that, if extant, they are to be found in just one place.[33]

However, if the search of documents corroborating the transmission hypothesis has not to be a random one, a typology of the messages and documents becomes decisive. Owing to the great distances and the yearly rhythm in the Europe-Asia navigation, Jesuit messages were standardized. Roughly, the following classes of documents may be distinguished:

a) the *annuae literae* (yearly letters): a document sent by every Jesuit province, with accounts of the state of things and the main events in every establishment there;

b) the *catalogi breves* or *triennales* (sent, respectively, every one or three years), giving a place by place list of the missionaries in that province and specifying their present occupations and their careers until that moment, together with a judgement on them by the superior and the patrimonial state of the province;

c) relations on special topics by the superior himself;

d) an accompanying letter to a, b, c, written by the same superior;

e) relations or letters sent by individual Jesuits to superiors in Portugal or in Rome;

f) private relations or letters sent by individual Jesuits to private persons (Jesuits and not) in Europe.

It is easily seen that: documents of the a) and b) genres were deprived of any scientific content, regarding the life of Jesuit institutions as such; those pertaining to c) and d) never dealt with technical matters (also because superiors usually ignored them); those pertaining to e) also could not deal with technical subjects, because superiors were not interested or familiar with them, but also because, for an assessment of a country's doctrinal traditions to be helpful in orienting the superiors' missionary "politics", it did not need to be very precise about disciplinary questions.[34] Thus, a search must be focused on documents of the kind f), because a Jesuit scientist would transmit scientific data on private messages only, so far as he addressed them to specialists, both within the Society and outside it.

To understand how they may be traced, some features of the Jesuits' planetary communication system must be kept in mind. Once a year (or two or three times, when more routes in addition to the Carreira da India were used), all the correspondence from Asia was collected in Goa (or Cochin) and directed to Lisbon in just one package (or one for every province); in Lisbon (or Coimbra) it was divided according to destinations. Thus, for instance, a letter sent to a little town in southern Italy was united in a package with other Jesuit correspondence from Portugal to central and southern Italy, and sent by ship to Rome (or perhaps Naples); finally, it was forwarded to the town and person it was addressed to. So at every halting point during the travel the package was opened and divided into its constituents (individual messages or groups of them), each of which had to be examined – and perhaps read – in order to ascertain were to send it. For this reason, a message could leave traces – or even a complete copy – along its way, and this makes casual, unpredictable findings possible.

However, the most natural possibility is, of course, that a private writing in a library or archive placed in its final destination; but how may this place be known in advance? If the author of a private scientific letter or essay is known, the number of potential addressees is circumscribed by his biography: usually a missionary left Europe when still a young man, so in most cases he had been trained mathematically in only one school, and his scientific acquaintances were few (in most cases only his master – or masters – and fellow students). Since both the masters and fellow students may be ascertained in the Society's catalogues, a search for their correspondence is – theoretically – a relatively circumscribed task. In practice, however, things are not so simple; during the years, both masters and students could have been (and usually were) destined to places other from that of the original school, sometimes very far from it; in addition, many Jesuit archives and libraries have been destroyed, were mixed with others or moved to other towns. More basically, all this simply does not work if the writer's name is unknown. Even in this case one may conjecture that, since the author was a mathematician, he had been formed in one of the few Jesuit colleges in Europe where a "mathematics academy" existed before 1630/40; but then the number of potential addressees grows much, because a certain document could have been written in an undetermined number of years.[35]

So, even in the case that transmission occurred and documents of it survive, it is no wonder that, as Almeida and Gheverghese Joseph frankly admit in the paper included in these proceedings,[36] until now the search did not meet with a single piece of evidence. As their report clearly shows (and was perhaps unavoidable at a first stage), their search was focused on collections, not on persons: namely, it was directed to the best known and largest European collections of Jesuit documents from Asia, not to others where the correspondence and writings of single potential authors of mathematically relevant reports could be. Therefore, the probability of useful findings could not be high. If the search continues, a change of focus is probably necessary; one may begin considering the school contexts of the 19 men listed, finding the names of their masters and schoolfellows and, after reconstructing their careers,

conjecturing which archives or libraries could preserve writings and correspondence sent to them.[37]

Obviously, this is not the only "interface" involved in the transmission process. A second one is to be postulated between the addressee and some of the non Jesuit scientists whose work was at the origin of the new XVIIth century mathematics and - according to the transmission hypothesis - could have been influenced by some Jesuit's reports. As known, these men were few; their papers, correspondence and manuscripts have been thoroughly investigated and, again, no single piece of evidence of a decisive influence coming from outside – or of a discontinuity which makes such an influence a necessary hypothesis - has ever emerged. So, unless also this kind of evidence is provided, a Jesuit document may prove that some important information reached Europe, but this would not be a proof of the fact that developments in Europe were an effect of Indian ideas.

One may conclude remarking that something like a '*counter fact*' exists. Had such a document reached a Jesuit mathematician in an European college, he would have exploited its potential himself or – if not sufficiently interested or able – would usually have communicated it firstly to other Jesuits. During years 1610 to 1650, the Society had in Europe such distinguished "pure" mathematicians as Grienberger, Grégoire de Saint Vincent, Paul Guldin, André Tacquet and some others. It cannot be assumed easily that the addressee of the document preferred to share its content not with his fellows but with someone outside the Society (not to say a Protestant). Yet, no trace of the Indian mode of computational mathematics is found in the works of representatives of the best Jesuit scientific schools all over Europe (Prague, Ingolstadt, Antwerp, Paris, Rome, Madrid's Collegio Imperial); on the contrary, most of them remained faithful to the classical geometrical mode in front of the new approaches (Cavalieri's method; uses of algebra in geometry, from Viète to Ghetaldi; analytic geometry). Obviously, this cannot be considered as a sort of negative evidence that proves the hypothesis to be false; but, unless positive evidence is found, it is an element against its plausibility.

ENDNOTES

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1 The basic reference work on Jesuits in Japan (Joseph F. Schütte, *Introductio ad historiam Societatis Jesu in Japonia*, Rome 1968), pays little attention to the scant evidence concerning scientific teaching and activities. Some cartographic work apart, only two aspects are known sufficiently: Pedro Gomez's lessons "On the sphere" to Japanese pupils (ca. 1593), conserved in manuscript and now published with two other groups of lessons by him (*Compendia. Compiled by Pedro Gomez, Jesuit College of Japan*. 3 volumes, Tokyo 1997-1999); a few astronomical observations and some scientific conversations referred in letters of some Jesuit missionaries, one of them being Carlo Spinola (1564 - 1622), the only competent mathematician in Japan's missions before the Society's expulsion and even after it (see, for instance, Henrique Leitão and José Miguel Pinto dos Santos, "O *Kenkon Bensetsu* e a recepção da cosmologia ocidental no Japão do séc. XVII", in *Revista portuguesa de filosofia*, 54 (1998), pp. 285-318). For more general – but also vague - evaluations see: Shigeru Nakayama, *A history of Japanese astronomy: Chinese background and Western impact*, Harvard 1969; Willi F. Vande Walle – Kazuhiko Kasaya (eds.), *Dodonaeus in Japan: translation and the scientific Mind in the Tokugawa period*, Leuven 2001.

2 For a general bibliography on Jesuit missions in India and other Asian countries see Laszlo Polgár, *Bibliographie sur l'histoire de la Compagnie de Jésus*, 6 vols., Rome, 1981-90, and Laszlo Polgár, Nicoletta Basilotta, "Bibliography on the history of the Society of Jesus", in *Archivum historicum Societatis Iesu*, 70 (2001), 140, pp. 265-579.

3 A recent collection of studies on Macao's college is John Witek (ed.), *Religion and culture. An international symposium commemorating the fourth Centenary of the University College of St. Paul*, Macau-San Francisco, 1999. It is, however, largely unsatisfying as for the scientific aspect, which has been the subject of a paper presented by the present author in a Tokyo colloquium in August 2005 (*The Macao college of the Jesuits as a meeting point of the European, Chinese and Japanese mathematical traditions*), to be published in the colloquium's proceedings.

4 Abstracting from some lines or pages devoted to the subject in general histories of science in India, a few relevant titles are: J. McFarland, "Jesuit geographers of India, 1600-1750", in *New Review*, 12 (1940), pp. 496-515; Mohammad R. Ansari,

Introduction of modern western astronomy in India during 18-19 centuries, N. Delhi 1985 (pp. 10-15 deal in a highly vague way with centuries 16-17); R.K. Kochar, "Secondary tools of Empire: Jesuit Men of Science in India", in *Discoveries, Missionary Expansion and Asian Cultures*, edited by Teotonio R. de Souza, New Delhi 1994, pp. 175-183; Edouard René Hambye, "Scientific activities and works of the Jesuits", in his *History of Christianity in India: eighteenth century*, Bangalore 1997, pp. 420-430 (poor as for the preceding centuries). Some studies of Jesuit cultural activities and education in southern India are also notable for an almost total lack of attention paid to science: P.S. Varde, *History of Education in Goa*, Goa 1977; Joseph Wicki, "La formazione della gioventù indo-europea a Goa", in Enrico Fasana, Giuseppe Sorge (eds.), *Civiltà indiana ed impatto europeo nei secoli XVI-XVIII*, Milano 1987, pp. 47-60; Gregory Naik (ed.), *Jesuit Education in India*, Anand 1987; Joseph Velinkar, *Jesuit education and enculturation in XVI century Goa*, in Anand Amaladass (ed.), *Jesuit presence in Indian History*, Anand 1988, pp. 66-77; Charles J. Borges, "Jesuit Education in Goa (16th-18th centuries)", in Prakashchandra P. Shirodkar (ed.), *Goa: Cultural Trends*, Panaji 1988, pp. 154- 164; Rui M. Loureiro, "O descobrimiento da civilização indiana nas cartas Jesuítas (século XVI)", in Berta Ares Queija-Serge Gruzinski (eds.), *Entre dos mundos. Fronteras culturales y agentes mediadores*, Madrid 1997, pp. 299-327; Ines G. Zupanov, *Disputed mission. Jesuit experiments and Brahmanical knowledge in Seventeenth-century India*, New Delhi 2001. One may also observe that studies on Matteo Ricci's years in Beijing alone outnumber by far all those on Jesuit cultural (not just scientific) activities in XVIth and early XVIIth centuries in India, including those on such important figures as Roberto de' Nobili or Giacomo Fenicio.

5 The very few exceptions, all published during the last five years, are mentioned in the following .

6 For the Portuguese colonies and establishments this is documented in Ugo Baldini, "The Portuguese Assistancy of the Society of Jesus and scientific activities in its Asian missions until 1640", in *História das ciências matemáticas. Portugal e o Oriente*, Lisboa 2000, p. 49-104, in part. p. 50-51. As for India, the only case of a specialist in mathematics who was sent to missions when his superiors already knew him as such is J.C. Gall: see table I and note 30. Antonio Rubino, the only other scientific figure who was in India for a long time before 1650, was designed for the missions before qualifying as a mathematician; he received an advanced course in mathematics by C. Grienberger in Lisbon, while waiting for his ship to Goa: see Ugo Baldini, "As Assistências ibéricas da Companhia de Jesus e a actividade científica nas missões asiáticas (1578-1640). Alguns aspectos culturais e institucionais", in *Revista portuguesa de filosofia*, 54 (1998), p. 231 n. 101.

7 Rubino (whose position is considered in the following) apart, only Giacomo Fenizio (1558-1632), a missionary in India since 1583, seems to have tried to use astronomy as a "neutral" subject in front of the Brahmins' suspiciousness: see Francis X. Clooney, "Roberto de Nobili's *Dialogue on Eternal Life* and an Early Jesuit Evaluation of Religion in South India", in *The Jesuits. Cultures, Sciences, and the Arts 1540-1773*. Edited by John W. O' Malley, S.J., Gauvin Alexander Bailey, Steven J. Harris, T. Frank Kennedy, S.J., Toronto 1999, p. 413 n. 2. However, Fenicio's was an individual attempt, with no relevant effect; moreover, his competence in mathematics was very limited and his work, the *Livro da seita dos indios orientais* (only published in 1933), has important descriptions of Indian rites and theology, but none of Indian scientific ideas.

8 This does not mean that all episodes of these research – even those prior to 1650 - have been described, nor that only astronomy and cartography were concerned. An interesting, although not-mathematical, subject were the measures of magnetic declination made both in navigation and on land, from Lisbon to Japan, with the hope that a recognizable pattern in its variations could be instrumental for measuring longitudes. Lists of the measures taken by different missionaries in different places (including Malabar) and times, were published by Athanasius Kircher and Giovanni Battista Riccioli after him (see for instance Kircher's *Magnes, sive de arte magnetica*, Rome 1654, pp. 315-6, 329, 348). Another, less known mathematical subject was natural history in the broadest sense (zoology, botanics, mineralogy, weather conditions). A notable example was the attempt by Johann Schreck (1618-1619) to produce a *Plinius indicus*, that is, an encyclopaedia of the natural history of the Indian region, which was written at least partially and is now lost (see Isaia Iannaccone, *Johann Schreck Terrentius*, Napoli 1998, pp. 49-57).

9 As known, Joseph Needham and others after him have maintained that China's science was much more advanced than the Jesuits' (and many western scholars' after them) descriptions of it, and that in some areas it was not below contemporary European science.

Even these students, however, did not maintain that such a science could evolve into modern science without a western intervention, nor that it exerted a dynamic role inside European science (if not for other reasons, simply because the westerners' knowledge of it was extremely partial).

10 Since this is the subject of some papers in this symposium, there is no need for quotations and bibliographical references.

11 As it is hardly necessary to recall, such a line of transmission has been asserted – as a possibility at least – in George Gheverghese Joseph's *The Crest of the Peacock* and in the same author's and Dennis Almeida's subsequent papers. Again, this being the main subject of this symposium, I feel exempted from giving a more detailed description.

12 The general suppression by Clement XIV came in 1773, but in Portugal and its overseas territories the Society was suppressed in 1759, and in Spain and France in 1768. Since all the missionaries in Asia belonged to these three Assistances of the Society, whatever their national origin, this meant that the Society's official presence in the continent ceased in 1768 (in Kerala in 1759).

13 A list of all Jesuits who left Europe for Asia is in Joseph Wicki, "Liste der Jesuiten-Indienfahrer 1541-1758", in *Sonderdruck aus Portugiesische Forschungen des Görresgesellscahft. Erste Reihe. Aufsätze zur Portugiesischen Kulturgeschichte*, 7 (1967), p. 252-450. Wicki registered all those who sailed to Goa, so including also missionaries who later reached the Far East or East Africa; in principle, as will be explained later, also many of the latter could be considered as potential agents of the transmission, but only in a few cases this potentiality could be a reality.

14 The end of the Roman primacy roughly coincides with 1636, the death year of Christoph Grienberger, Clavius' main assistant and successor. In the years following it gradually passed to colleges in the German Empire, France and Belgium.

15 An examination of the Jesuit students of Sanskrit only mentions de' Nobili for years prior to 1630 (Anand Amaladass, "Jesuits and Sanskrit studies", in Teotonio R. de Souza & C. J. Borges (eds.), *Jesuits in India: In historical Perspective*, Macau 1992, p. 212-213). Rubino's letters show that, during more than 30 years, he did not succeed in convincing a Brahmin to share his learning with him. It has been asserted that in his reports "there is little evidence of a major input from a native informant", and that they do not "summarize the contents of any recognizable native work" (Joan-Pau Rubiés, "The Jesuit Discovery of Hinduism. Antonio Rubino's Account of the History and Religion of Vijayanagara (1608)", in *Archiv für Religionsgeschichte*, 3 (2001), p. 210-256, in part. 221-222).

16 See the "Rules of professors of mathematics" in Laszlo Lukács (ed.), *Monumenta paedagogica Societatis Iesu*, V, *Ratio atque institutio studiorum Societatis Iesu*, Rome 1986, p. 402.

17 For data documenting this situation and a chronology of the teaching of mathematics in the Iberian states see: ....

18 Sometimes, the scientific work performed by a few Iberian missionaries performed scientific work has been considered as documenting an extended knowledge. This, for instance, has happened for the measures of the positions of many Indian and Afghan towns taken by António Monserrate (1536-1600) during his travel to reach Akbar's court and during another to Afghanistan in the Emperor's retinue, or for the astronomical handbook written for a Chinese public by Manuel Dias junior (see note 25). However, both the topographical measures (taken with the methods and within the limits used by Monserrate and other Jesuit travellers, as distinct from the Society's mathematicians in Asia, who used more refined astronomical methods) and the compilation of an elementary "sphere" were possible to a cultivated person with no institutional training other than the basic curriculum. On the contrary, this was not the case for a full understanding of the refined kind of calculations found in Kerala mathematics.

19 For Portugal and its missions such reproaches were repeated until 1692, when General Tyrso Gonzales dictated strong orders that obliged the Portuguese Provincial to institute both elementary and advanced mathematics courses in the country's main colleges: see Ugo Baldini, "The teaching of Mathematics in the Jesuit Colleges of Portugal, from 1640 to Pombal", in Luis Saraiva and Henrique Leitão (eds.), *The Practice of Mathematics in Portugal*, Coimbra 2004, in particular pp. 315-327; see also pp. 648-664, 666-670, 684-688, 695-698, 704-723.

20 The sign "*" refers those who had attended an advanced course on mathematics (the "academy of mathematics") in Rome; "#" to those who had attended such a course in another college of the Society (always in Italy, Germany or Bohemia); no sign is apposed to persons with just a basic training in mathematics, who learnt something more from some of their fellows while travelling to India or after reaching it; names underlined are those of the Jesuits

for whom India was the definitive, or the most lasting, country of mission (only 3 in a total 19)].

21 While going back to Europe.

22 While going back to Europe.

23 Ruggieri (1543-1607) is included here only for the sake of completeness. Not a specialist, he acquired some mathematical knowledge as Ricci's collaborator during some of the years he spent in China before returning to Italy (1579-1588), but this was after his first sojourn in India (nothing is known about the second), and what he learnt certainly was not algebra and related subjects. His best known "mathematical" work is a map of the Chinese empire, perhaps the first complete by an European; see *Diccionario histórico de la Compañía de Jesús*, Rome-Madrid 2001, vol. IV. Rubino, Gall and Uwens are shortly discussed in the following. On Ricci no information or bibliography is needed (see the synthesis in *Diccionario*, vol. IV). Dias (1574-1659, not to be confused with an anonymous Jesuit who was his uncle and was also in the Asian missions) is the only possible exception to the general absence of the Iberians. His case is analogous to Ruggieri's, because formally he did not study mathematics while a student in Coimbra's college, but mastered some of it perhaps attending for a short time a course by Grienberger in Coimbra, that of Rubino in Malabar (see table II, at the year 1603-4) or another by Aleni or Sambiasi in Macao. His best known (or, rather, only) scientific work was a handbook of spherical astronomy written for a Chinese audience (see note 20). On him, and all those following, basic biographical information and bibliography are provided partly by the same *Diccionario* and by Baldini, *The Portuguese Assistancy*, pp. 84-88.

24 Portuguese ships coming from Lisbon usually sailed along Mozambique's coast until north of Madagascar, then turned north-east toward south India. So missionaries destined to places in Africa's coast from the Cape to Mozambique landed before, while those directed north of Mozambique first reached Goa, then sailed on ships directed to ports in Kenya or Somalia.

25 A well known case are the observations of the Comets of 1618 made by Schreck, Kirwitzer, Bell and Rho during their navigation to Goa and also there, together with Rubino. They were reported in a letter written by Kirwitzer and sent to the Jesuit mathematicians in Ingolstadt; these, in turn, made it known to Kepler, who published the text with his own observations (*Observationes Cometarum anni 1618 factae in India Orientali a quibusdam Soc.* *Jesu mathematicis in Sinense regnum navigantibus*, Aschaffenburg 1620). Ricci's interest in the Indian calendar is explained by the fact that he left Rome for Lisbon in 1577, the year in which the Pontifical Commission for the reform of the calendar published its project for a reform (finally introduced in October 1582). The best technician in the commission was Ricci's master, Clavius, who was interested in Asian calendars as suggesting solutions for some of the problems faced by the commission (see George V. Coyne, Michael A. Hoskin and O. Pedersen (eds.), *Gregorian Reform of the Calendar*, Rome 1983.

26 See his letter to his master Christoph Grienberger quoted in the paper by D. Almeida and G. Gheverghese Joseph in this volume. Among those who remained in India for some years, Matteo Ricci, whose letters and papers have been carefully investigated, never mentioned such an access both to his master Clavius or to the Roman superiors.

27 The list of Rubino's papers in Carlos Sommervogel, *Bibliothèque de la Compagnie de Jésus*, VII, cols. 279-80 (additions in vols. IX, X, XI, XII) is not complete. For others, and for a bibliography, see Rubiés, *The Jesuit discovery*, notes 1 and 5.

28 For the few known data and bibliography on Gall (1586-1643) see Ugo Baldini, "L'insegnamento della matematica nel collegio di S. Antão a Lisbona, 1590-1640", in *A Companhia de Jesus e a missionação no Oriente*, Lisboa 2000, p. 286-87.

29 Uwens (1618-1667) is ignored in Sommervogel's repertory. For essential biographical information see H. Hosten, "Fr. Henry Uwens, alias Henry Busi S.J., a Missionary in Mogor (1648-1667)", in Examiner, 68 (1917), pp. 407-9, 418-20.

30 The best known (not the only) case of mathematics teaching during navigation is that of Schreck (1618). Mathematics was certainly taught in Macao in the years 1610 to 1613 by Aleni and Sambiasi and in 1617-18 by Wremann.

31 It has to be recalled that the final phases of the Jesuit curriculum were constituted by a three years course in philosophy (logic, natural philosophy, metaphysics) and a four years course in theology (to which only a part of the young Jesuits was admitted). According to the official curriculum in the *Ratio studiorum*, mathematics was a discipline in the former, being taught during the second year by a professor different from that of natural philosophy. The course in theology included such disciplines as scholastic (i.e., fundamental) theology, moral theology, Holy Writings and (sometimes) polemic theology (anti-protestant

polemics) and Hebrew. Owing to lack of adequate lecturers and a sufficient number of students in these colleges (as well as in Macao's), in some years – particularly before 1610 or so – courses were suspended, shortened or deprived of some disciplines. As for the philosophy course, the Society's yearly "catalogues" usually mention the part of it being taught in a certain year using the name of that part: logic, natural philosophy (simply labelled as "*philosophia*"), metaphysics. Sometimes, however, the word "*philosophia*" extended to metaphysics, or (more rarely) generically to any of the course's disciplines. In the Latin of the catalogues and other Jesuit sources the word *mathematica* (or *mathesis*) always refers to the teaching in the second year of philosophy; on the contrary, *arithmetica* is used only as a name for the primary teaching (regarding boys from 7 to 10/12 years), which associated elements of reading, writing and catholic catechism with basic arithmetics (little more than the four operations). Usually, the lecturers of arithmetics were not those of the mathematics course. The years missing are those for which no information has been found so far. Sources: Archivum Romanum Societatis Iesu (ARSI, the Society's central archive in Rome), *Goa* 24, *Goa* 25; Joseph Wicki (ed.), *Monumenta historica Societatis Iesu. Documenta Indica*, 18 vols., Rome 1948-1988

[32] This is an approximated, not a rigorous, picture because even in the case of India-Europe communications the confrontation between the Portuguese and the Dutch in the Indian and Atlantic oceans sometimes suggested to mail a same message on more ships travelling on different routes (all of them basically pertaining to the Carreira).

[33] An example of this is a document discovered during D. Almeida's and G. Gheverghese Joseph's research: namely, a relation to the General in Rome by a Goa Provincial, concerning the astronomical and mathematical knowledge of the Indians conserved in the ARSI codex *Goa* 58 (see, in this volume, their *A summary report on the investigation on the possibility of the transmission of the medieval Kerala mathematics to Europe*, note 29). If interesting, being the only document of the sort known so far before middle seventeenth century, it contains just generic data and evaluations.

[34] This and other difficulties made that even a volume specially devoted to investigating the European circulation of Jesuit letters from Asia was able to produce general lists of those official and, among those private, of the printed ones (like that of Kirwitzer: see note ....), but produced almost no result for the manuscript ones:

see John Correia-Afonso, *Jesuit letters and Indian history, 1542-1773*, second ed., Bombay 1969.

[35] See the final remarks in Almeida and Gheverghese Joseph, *A summary report*: "At any rate what we do know from the project study is that the null hypothesis of no direct transmission is sustained by the evidence gathered [up to now]".

[36] Some more than conjectural information already exist. For instance, specific deposits of manuscript papers and correspondence of members of the Ingolstadt school, decisive for Gall, have been found; see, for instance, August Ziggelaar, "Jesuit Astronomy North of the Alps", in Ugo Baldini (ed.), *Christoph Clavius e l'attività scientifica dei Gesuiti nell'età di Galileo*, Rome 1995, pp. 101-132.

DISCUSSION

The paper essentially argues against the possibility of any transmission of Kerala mathematical ideas to Europe without disputing the fact of priority of some of the Kerala discoveries. The argument is based on showing a lack of motivation, opportunity and evidence (both circumstantial and documentary) for such a "hidden migration" of ideas. But this would seem contrary to the evidence produced in the work of Almeida and Joseph (references to which are found in Endnote 1 of the Introduction and the paper by Almeida and Joseph in this volume) to indicate that Jesuits had the motivation and opportunity to learn of these ideas, that the Brahmin elites were neither reticent nor the only class to have this mathematical knowledge. The paper did not address these alternative claims.

However, the paper recognizes that if we seek documentary evidence then it is necessary to take a "focussed" approach since the Jesuit communications from the East went through different routes, to different colleges, and to people continually travelling to different places in Europe. Therefore, the recommendation that one identifies networks of teachers and students trained in mathematics, and look for literature that might indicate communication in places where they resided, identified through the Jesuit yearly catalogues, would make the search for documentary evidence more tractable.

The paper not only attempts to explain why, prior to 1650, there has been no known cases of *scientific* exchanges between Europe and India, but also to examine the focus of future research needed to find documentary evidence for these exchanges (if they had occurred). However, the reasons given for the lack of documentary evidence would make it unlikely that one would find such evidence in the future. For example, even in the case of China, those most familiar with its mathematical and scientific doctrines, such as Matteo Ricci and those who succeeded him in Beijing's mission, concluded that China could provide nothing that could meet Europe's scientific standards. The Jesuit judgment concerning India was likely to be even harsher. However, it is difficult to be comfortable with the

idea that these Jesuits who were aware of the work of their mathematics teacher, Grienberger, in the art of technical computational arithmetic transmitted by the Arabs from India, would have had this 'harsh' assessment of Indian science.

A conclusion of the paper is that the Jesuits would have had little motivation to seek mathematical knowledge in India. But set against this argument, we know that Ricci was instructed to find out about the little known science of India for the benefit of Peitro Maffei, the historian of Jesuit expeditions abroad*; Further, apart from Rubino there were no Jesuits sent to India with the ability in both mathematics and the Indian languages to appreciate a tradition so different from the European. Even the few mathematically trained Jesuits who were sent East were directed towards China, given the perceived receptivity and importance that the Chinese attached to learning from the European mathematical astronomical tradition. But this view is contradicted somewhat by the example of the later Jesuit De Nobili who showed his versatility in writing a critique in Sanskrit of the earlier Indian astronomy/astrology of Varamihira's *Vedanga Jyothisha.*

A further argument relating to the lack of opportunity is that the colleges set up by the Jesuits in India in Goa and Cochin did not teach mathematics but only theology and philosophy. Hence they did not recruit or train people capable of interacting with and learning from the Indian mathematical astronomical traditions. This may be the case but the exception of De Nobili suggests that once immersed in the local tradition it may have been possible for Jesuit missionaries to learn knowledge formulated in a different epistemology.

* "(Ricci)...had resided in the cities of Goa and of Cochin for more than three years and a half (September 13, 1578-April 15, 1582): he had been requested to apply himself to the scientific study of this new and imperfectly know country, in order to document his illustrious contemporary, Father Maffei, the 'Titus Livius' of Portuguese explorations." Henri Bernard, *Matteo Ricci's Scientific Contribution to China*, Westport, Conn., Hyperion Press, 1973, p 38

** Vincent Cronin, *The Wise Man From The West,* London, Collins, 1984, p 178- p180)

Despite the pessimistic conclusion of the paper regarding transmission, it suggests one should focus on the networks of teachers and students trained in mathematics and their movements and communications which can be traced through the Jesuit Order's yearly "catalogues". Although it is not optimistic concerning the outcome of such a search, the paper advises a search among the undocumented private letters that the Jesuits sent to their friends/colleagues which might contain some mention of some indigenous knowledge of higher mathematics. But it may need a strenuous and time-consuming endeavour to identify these documents which would be akin to a search for a needle in the haystack.

13

UNCATALOGUED MALAYALAM MANUSCRIPTS IN EUROPE - A Report of Work Carried Out in Two Libraries in Rome

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This report is the written record of the work carried out by the undersigned at the Bibliotec Apostolica Vaticana and Bibliotec Nazionale Centrale di Roma in Rome from 1st to 20th May, 2004 as part of a project entitled, 'Medieval Kerala Mathematics: The Possibility of its Transmission to Europe', funded by Arts and Humanities Research Board (AHRB), U.K, for exploring the process of dissemination of the mathematical knowledge of Kerala to Europe. Spread of ideas of western culture and civilization to the east has attracted a large number of scholars but the influence of eastern knowledge systems on modern western intellectual advancements is still a dimly lit area. Recent enquiries into the problem have framed some strong hypotheses about the influential role of the *ganita* tradition of Kerala in bringing about some path breaking advances in the modern Western mathematical knowledge (Joseph, 1995; Almeida and Joseph, 2004).It has been maintained on the basis of reliable data that this transmission of knowledge took place through the Jesuit missionaries who were working in Kerala during the above

mentioned period. It is against this intellectual background that a thorough search becomes necessary for any clue in the form of direct evidence to supplement or enhance the existing circumstantial evidence for explaining the transmission of knowledge of mathematics from Kerala to Europe. Our aim in combing the missionary collections of palm leaf and paper manuscripts in the Libraries in Rome and Vatican was to examine whether they contain any direct evidence in the form of mathematical or astronomical texts.

The present search for Malayalam mathematical and/or astronomical texts in Europe was concentrated on two major collections: (i) Bibliotec Apostolic Vaticana (BAV), and (ii) Bibliotec Nazionale Centrale di Roma (BNCR). Both of them have a fairly good number of manuscripts. A detailed list of Malayalam manuscripts in these collections is given below:

Bibliotec Apostolic Vaticana

This collection consists of four series:

I.	Borgiani	74	(1-74)
II.	Vaticani	75	(1-75)
III.	Rossiani	2	(1190,1191)
IV.	Barberini	1	(109)

1. Borgiani series

1. Latin-Malayalam Dictionary
2. Malayalam grammar in Portuguese dated,12 .1.1733
3. (i) Proceedings of the synod of Diamper,1599
   (ii) Christian catechism
10. Malayalam – Portuguese Dictionary by Earnest Hanxleden
14. Prayers, dated 1746A.D
18. Synod of Cranganore of 1606 A.D.
19. Sanscrit dictionary in Malayalam letters.
20. Synod of Diamper.
21. Proceedings of the synod of Diamper and its History, dated, 25.1.1599

22. Catechism, dated 1767A.D.
23. Religious writing.
24 (This item is missing in the catalogue. Number23 is followed by no.25.)
25. Grammar of Malayalam language in Portuguese (number repeated) Malayalam dictionary
26. Christian Catechism.
27. Malayalam grammar in Portuguese.
30. Calendar for the year 1792A.D.
34. Religious writings of 1771 A.D.
35. Bhagavatam (Parts).
*37. *Kiratam Tullal.*
*38. *Bhagavatam kilippattu* (parts) The catalogues describe it as'*Chandrodayam.* or the nativity of Moon
*40 '*Somavaravratam'*, in *kilippattu* style (importance of the religious observation on Mondays. Collected by Fr. Paulinose with a note 'snaparvam'
*41. Parts of Bhagavata with commentary.
*52. Correspondence of the Varappuzha seminari.
*53. Correspondence
54. Religious work. Lives of the Apostles, St. Peter, St. Paul, St. John and St. Andre
*55. *Keralolpatti*, (Earlier catalogues describe it as Bhagavata]
*57. Commentary of Amarakosam in archaic Malayalam.
58. Religious work.
*60. *Misihacharitam Pana* (the story of Jesus Christ versified in the metre called 'Pana')
*69 Indigenous treatment for different types of fevers like vatajvara, *pithajvara*
*73. Parts of Uttararamayana and Satamukharamayana (Antony Vallavanthara describes them simply as poem related to Hindu religion').
74. Christian religious work.

2. Vaticani series

8 Malayalam grammar
12. Malayalam grammar

13. Malayalam grammar
14. (a) 1-107 Angamali synod of 1606
    (b) 108-239 Grammar of Malayalam.
15. Malayalam-Latin Dictionary
16. Roman and Malayalam Dictionary.Portuguese into Sanskrit and Malayalam.
17. Poruguese- Malayalam Dictionary
18. Proceedings of the Diamper synod
20. Political science. Kautalya's maxims.(Sanskrit)
24. Religion
*31.Christian religious work in Malayalam
35. Religion
38. Religion. Prayers etc.
40. Religious poem *Kristapuranam*

3. Rossiani series

*1190.Christian catechism.
*1191.*Samkshepavedartham* A religious work in Malayalam

4. Barberini Orientale

*109 Religious text. Proforma to be filled up at the time of ' *'jnanasnanam' or ' mamodisa'*

Bibliotec Nazionale centrale di Roma has Malayalam manuscripts in two sections, namely, Inventario folio Palma of palm leaf manuscripts and Scala Maria in Manoscritti Orientali collocati in Alti Fondi of paper manuscripts. What follows is a list of the two sets of manuscripts:

Folio de Palma

4. Ezhuthacchan's *Ramayanam kilippattu Yuddhakandam.*
8. Magic and medicine (archaic writing)
9. *Ramayanam kilippattu* Yuddhakandam.(as in no.4 above)
*10.Arnose Padre's 'Chodyaparvam' (1779 A.D) dated, 955 Dhanu 23 Copied by Kodiyan

Cheriyan Uthuppu dated, 955 Dhanu 23 (1779 A.D)
11. Cherusseri Namputiri's *Krishnagatha* (parts only)
*13.(i) 'Ashtangahrdayam'(parts only)
    (ii) Documents of transactions of land and money dated, 962 A.D.
    (iii) Documents of transactions of land and money dated, 964 A.D.
    (iv) Continuation of 'vidhiparvam'(a sargam in the *Misihacharitham Pana*)

Manoscritti Orientali collocati in Alti Fondi

Maria della Scala

1(140) Fra Paolino da S. Bartholomio (Giovanni Filippo Verdin.Miscellania Indica)
* (i) Alphabet including conjuncts
*(ii) Multiplication tables. (Place value not introduced)
*(iii) Malayalam .Formal expressions of social heirarchy
(iv) Latin portion
(v) Dictionary (Mogalanum Indostanum.)
*(vi)Amarakosam (parts)
(vii) Writing in Syriac
(viii) Description of topography of Malabar.
(ix) Religious writing.
(x) Malayalam Grammar (fragmentary)
(xi) Notes on caste, moeurs, costumes, sciences. In the last paragraph Fr.

Paolinos adds:

> " ....es Brahmes calculent les Ecclypses, et font des Almanachs, mais, a la cote Malabare ilya une Difference de dix jours entre nous et eux......"

8(65) **Miscellaneous. Dated 1782 A.D (This collection contains portions of Sanscrit and Malayalam poems collected by Fr. Paolinos and also poems composed by the author himself. The poems of the author are of**

inferior in literary quality, perhaps due to the reason that he was trying to learn the techniques of composition in the native style and for the reason they are worth noting from a cultural point of view.)

*(i) Magham (Skt. poem. some portions only)
*(ii) Stray slokas including one praising the Raja of Travancore.
*(iii) Own compositions. (Namam Paulinosadhipatiyam namamuyarnna sakhadgi....)
*(iv) A song in vanchippattu style which describes a journey of the Raja of Cochin with his minister Kompi Achan.
*(v) A poem in *Kalakanchi* metre and *kilippattu* style which addresses the kili(parrot) and requests her to sing "new" song. Subject matter of the 'new' song is the story of Jesus Christ
(vi) Writing in Latin.
*(vii) Lakshanam, meaning characteristic features of *devas*. Here the author quest questions the origin and divinity of the Hindu gods.
*(viii) The story of Mar Chresimma in the *keka* metre

11 (189) Malalyalam grammar
16(196) Grammar (points only)
17(194) Yudhishthiravijaya ,Sanscrit poem.(Transcribed by Ernesto Hanxleden)
22 Correspondence on Varappuzha [copied to CD]
25(246) Malayalam dictionary.
30(274) Mal- Latin Dictionary (Ernesto Hanxleden)
35-36(146-147):
*35 (i) Writng in Latin
*(ii) Grammar-some verbal forms only. Remaining portions contain writings in various languages like French, Latin, and Italian
*36 (i) Grammar
(ii) Theology
(iii) Flora of Malabar. The author gives names of some plants like vellattamara (white lotus), Manhattamara (yellow lotus),etc.and a few stanzas from the 'Dravyaguna',describing the qualities of herbs.

Gesuit 963 (3092) Grammar of Malayalam in Portuguese language.
Varia 52 (641) Latin – Malayalam Dictionary

Apart from the above centres, we had an opportunity for visiting a Jesuit library near Vatican. This institution has no Malayalam works. Nor does it have works pertaining to Kerala. But significantly, its Tamil works belong to the same tradition of the Christian religious catechism, and religious works etc. of the other collections of Rome and Vatican. Again, it is interesting to note that the same tradition is visible in some other collections of Europe such as the Bibliotheque Nationale de France (BNF) A list of Malayalam works in the BNF is given below (Cabaton : 1912)

Indien 728 A Grammar of Malayalam
737 Dictionary
752 Christian catechism
765 *Puthan Pana* (also known as *Misihacharitram*). Christian religious literature.
*766 Letters of local notables to the French in Mahe.(Written in *kolezhuthu* script, deciphered by the present writer)
767 Christian instructions regarding marriage.
768 Christian instructions regarding marriage.
769 The acts following the Synod of Kodungallur of 1606 A.D.
770 Totally illegible as the letters are not inked.
771 Indigenous remedies and magical formulas.
772 Indigenous remedies and magical formulas.
773 A lengthy text in *kolezhuthu* (78 leaves). Local knowledge of Astrology and allied topics Language is archaic Malayalam with profuse sprinkling of Tamil. The leaves belong to different *granthas* as some look very old with a smoky tint while some others look fresh with a golden yellowish colour. The deciphered portion contains the formula of the *nakshatra* constellations and their *kooru* positions
*777 Letter of Kurungottu Nair

778 Literature. *Ramayanam Kilippattu* (parts) Stories of Christianity and disparaging Hindu cults and practices.

780 Religious writing

*781 A single leaf with a fragmentary letter of which the beginning is lost. The tone of the letter with its customary style suggests that it was written by a person of high status, probably a royal personage.

796 A Persian – Malayalam Dictionary.

988 Malayalam alphabet and numerals (with place-value).

1002 Catechism of Roman Catholics.

*1027Stray leaves. Records of chitty and money transactions without any details.

[* The asterisk mark indicates the manuscripts identified/ deciphered by the present present writer]

These lists of manuscripts seem to suggest that their range is restricted to certain areas of intellectual interest of the western missionaries who were working among various colonial native communities in South India in general and Kerala in Particular. Apparently, the areas of missionary interest, as gleaned from the above lists, were Lexicography, Grammar, Religion, Literature, Tradition, and Science and technology. The trend of the evangelical groups becomes clearer when the collections of manuscripts and their subject-matter are put in a tabular form. [See the Table below]

	Dictionary	Grammar	Religion	Literature	Tradition	Corres-pondence	Science	Technical writing
Barbereni			109					
Borgiani	1,10,19,57,64	2, 25, 27	3, 14, 18, 20, 21. 22, 23, 28. 34, 53, 54, 58 60,71,74	35,37,38,40, 41,56,60,73	55	52, 53	69	
Vaticani	15.16.17	8,12,13	14,18,19,24,31 35,38,40	N68			20,32(b)	
Rossiani			1190, 1191					
Folio palma				4, P9, P11			13	
Maria de Scala	35	4, 8, 11, 12,30,34, 35,41	1, 2, 10, 22(i) 36	7		22	8, 9	5
Bibliotheque Nationale France	737, 796	728, 988	767, 768, 769 778, 780, 1002	765, 778		766, 777	771, 773	771

Malayalam Manuscript Collections and their Subject Matter

The above table is suggestive of an overarching paradigm which determined the character of the collections. One of the repetitive items among the manuscripts are 'Dictionaries' that link foreign and native languages such as Sanskrit, Tamil, and Malayalam, Latin and Portuguese. This was an essential primary step for those who seek entry into the interior of a tradition. It is also necessary to learn the language of the native people to work among them, to convert them and to educate them. In the large spectrum of this lexicographical tradition we can see several types including brief, functional texts as in the case of some items in the 'Maria d scala' collection of BNCR as well as scholarly and pedantic larger texts such as the Malayalam – Portuguese dictionary of no.17 of the Vaticani series (BAV) above. Thus the dictionaries are in all probabilities part and parcel of the existing paradigm of evangelism. Compilation of dictionaries was only one aspect of this evangelical project. Apart from dictionaries, the missionaries compiled also proverbs and other sayings taken from well known works, including classics. The compilers very well knew that ideas are effectively expressed in the traditional lore and hence the special interest in items such as proverbs, folk narratives in prose as well as verse such as Paulinos Padre's *Adagia Malabarica*. However, they do not appear in our collections. Conventional explanation of such activities is that the missionaries were great lovers of language and culture and that their aim was to enrich the native languages with all kinds of necessary linguistic tools including lexical materials. From a functional point of view the lexicons were compiled by the missionaries with a view to helping their own colleagues to achieve their religious goals and this need not necessarily underrate enthusiastic scholarship and linguistic skill of the learned missionaries.

Grammar is another area of missionary interest and usually grammatical works of missionaries also are explained away as pioneering yeoman service to native languages. The service aspect apart, they had an important function to carry out in equipping foreign priests with local dialects and their correct usage. Various aspects of local dialects and regional languages

are compiled, codified and published by the missionaries in order that linguistic skills of the preachers of the new faith might be enhanced. Thus the missionary grammar had a significant role in the religious conversion of the native population and it fits well in the general shift of paradigm of that age. This shift of paradigm is generally characterised by a trend of modernisation which was not without serious and all-pervading repercussions in political, social and cultural aspects of native life. Pioneering works in what is usually called missionary Malayalam were naturally and necessarily in prose of the day-today use of the common people and this popular use placed the native prose in a privileged position. Prose was considered in the enlightenment modernity as the vehicle of thought against verse of emotional content and that was why the position of prose was held high. Interestingly enough, this higher status of prose was appreciated as a characteristic feature

Understandably, the largest section in the collections is that of religious literature: 37 out of 92. This corpus includes works on biblical themes, Christian catechisms, and also the acts and proceedings of the synods of Udayamperur and Angamali. Some works such as *Misihacharithram* that is now included in this section may fit well in the section of literature also. Interestingly enough, the authors of Christian religious works, at least the missionaries, have claimed that they were introducing 'new' subject matter into the native literary tradition by bringing in Christian myths and legends. In an important work on the life of Jesus Christ, the author addresses the parrot in the traditional *kilippattu* style and requests her to sing 'new' stories [Maria della Scala,8(65) Miscellaneous [v] above]. One should not fail to notice that this early trend of the 16th and 17th centuries of following the native literary tradition with admiration and scholarly interest to introduce a new subject and faith was replaced by a later trend of 19th century of criticising and rejecting the native tradition as entertainer and sinful and therefore inferior to the modern western (*vilathi*) style, which was according to the missionaries, more suitable for serious intellectual exercises (Gundert,1842).

Literary works of native tradition such as *Ramayanam kilippattu, Krishnagatha* etc. had a double role to play in the

missionary writings. On the one hand, they were models, as we have noted above, for the Christian writers to be adopted in composing works on 'new' themes. One of the famous and popular Christian work, *Misihacharithram* takes, at least ideally, the traditional *Pana* as its model and hence its more popular name *Puthanpana* (Indien,765 of Cabaton,1912). Fr.Paulinos has composed poems employing Christian themes and they depend on traditional style of diction and narration of Malayalam poems [8(65)viii]. On the other, many of them were shown as examples for heretical works.

Several native traditional works and their contents are put under fire to show how they propagate false and heretic faith which is contrary to the 'high' Christian ideals. A portion in a theological work composed by Fr. Paulinos compares native deities and the Christian God to show how the divinity of the Hindu gods becomes questionable when their origin is examined carefully [Maria della Scala, 35-36 (146-147)]. Severe criticism of alien faiths is a necessary step to assert and preach one's own religion in a context of large scale proselytisation. Hermann Gundert seems to follow Fr.Paulinos when he was writing his disparaging *nalacharithasarasodhana,* a detailed critique of the 'Nala' episode in the Mahabharatha, (Gundert in Zacharia, ed.'92:81-120) to criticize some Hindu beliefs and divinities. The presence of malayalam literary works in the manuscript collections under discussion is better understood when it is examined against this historical and cultural context.

Historical tradition is the least represented category of records in our collections. We have only a single text of *keralolpathi* in the whole of the collections in Rome to represent the category of historical tradition [no. 55 of Borg. Ind]. At the time of the advent of the western religion, there were at least three native modes of perception of the past. The most popular among them was the *Keralolpathi* tradition that sought to legitimise the social, economic and political status of brahmans and the ruling houses of Kerala. There was a Muslim historical tradition of the *Tuhfath-ul Mujahideen* and it was a response of the Muslim traders on the Malabar coast when their interests were greatly disturbed by the western monopolist mercantilist groups. Apart from these two, there was another tradition of a

folk historical consciousness that preserved those past events, the society thought to be relevant for perpetuation. This is usually in the form of songs that become popular very soon. Generally, they are composed on local themes like petty skirmishes, village outbreaks etc. Occasionally they resorted to themes like the famous twelve-yearly *mamankam* fight that took place at Thirunavaya or the death of a *chaver* or suicide squad at the *mamankam* or the death of a yet another hero for the sake of his patron.. Such songs had a steady demand during local festivals of different communities as well as in the weekly markets that were held regularly in the countryside. It is against this native cultural context that the acute scarcity of historical traditions become conspicuous. A possible reason for the lack of missionary interest in historical traditions could be the fact that none of the existing traditions was in support of the concerns of those who collected the native records. In other words, being familiar with another form of historical narrative, those who collected these manuscripts were selective and they were looking for the materials that supported their programme of proselytisation, directly or indirectly. Further, it is also not improbable that the missionaries could not understand the real value of the traditional historical accounts of native cultures. A historical event of interest of the missionaries was two synods of Udayamperur and Ankamali and/or Kodungallur dated,1599 and 1606 respectively. It seems to be noteworthy that these incidents including the acts as well as proceedings are recorded with great care.

Official communications between the Christian agencies and the local authorities form part of the section of Correspondence in the present manuscript collections. This category includes a number of letters between the government of Travancore and the Christian missionaries of the Varappuzha Seminary [Maria delle Scala, 22]as well as a few letters between some of the local authorities and the French officers at Mahe, the then French factory on the western coast. Regarding Varappuzha, the letters are of economic as well as cultural importance since they are directly connected with the establishment of an institutional base for the programme of proselytisation. Here it may be remembered that Varappuzha is the first Christian centre in

Kerala to have a printing press. Thus, the place is often identified with the starting point of modern age in Kerala. In addition to officially sanctioning missionary activities among the native communities, this corpus of records went a long way, as cultural capital, in legitimising the social status of the foreign missionaries. To put it in other words, though outwardly it may appear to be official and secular, the whole set of records belongs to the general paradigm of Christianisation and modernisation. Here it may be remembered that Varappuzha is the first Christian centre in Kerala to have a printing press. Their religious nature cannot be overlooked. The correspondence between the local magnates of Mahe and the French authorities is directly linked with the western colonial interests and strategies. Thus, these records acquire added importance in the overarching paradigm of colonialism.

The last two sections of science and technical writing of our table can be taken together since they are closely connected with each other. Representation of these categories is not so meagre as in the case of historical tradition, but the contents of the available manuscripts are of a primary level. Generally speaking, major chunk of the subject matter of scientific and technical writing is restricted to the area of health care and calendrical knowledge. Even in the lessons for beginners, arithmetic portions are scanty, not to speak of higher mathematics. Geography makes its appearance only sporadically and that too is with reference to the 'epico-puranic' tradition [no.67of Vat. Ind]. The only exception to this is a description of geography of some real places of India as well as some astrological writings about the *rasi,* divisions and their tutelary planets [no32(b) of Vat Ind].There is a brief attempt to describe the flora of Malabar [no 36 of Maria d scala] and in the available portion there are only names of some medicinal plants and their equivalents in Latin .It may be noted in this connection that BNF, Paris has a lengthy palm leaf manuscript [Indien,no.773] which contains some portions on astrology and astronomy, but for want of time, the present writer has not been able to decipher this work in full. The deciphered leaves inform us that they belong to different texts and that our knowledge of the content of this manuscript is fragmentary.

At an early stage of interaction with the native communities, European missionaries like John Ernestus Hanxleden (known to Malayalis as Arnos Padre) were eager to come in contact with learned people of higher sections of society. After an initial aversion of certain learned people of upper castes, Arnos Padre was successful in earning friendship of several knowledgeable Nambudiri scholars who were well versed in different branches of knowledge (Thomas, 1989:110). He is said to have collected some important texts of various disciplines from Trichur and its surroundings. In all of the collections of Rome, Vatican and Paris the works of Arnos Padre have a prominent place. It is not improbable that the manuscripts collected by him also must have reached the same destination but they are not traceable in those centres at present. Arnos Padre lived and worked in Kerala during the first three decades of the 18th century and quite probably he was following an intellectual trend set by his predecessors. By the time of Arnos Padre and his followers a shift of interest to the proselytisation of natives of lower social status is visible. This means that the people addressed by the newly created Christian literature were not learned sections of society but lower communities who were deprived of several privileges including higher education. Therefore, it is in vain to expect higher levels of knowledge from the missionaries or from the newly converted groups.

From the above description it becomes clear that the seemingly chaotic nature of these collections of manuscripts is deceptive, and also that there is an overarching paradigm which renders them meaning and significance. It is this overarching paradigm that contains the works in these collections belonging to various branches of knowledge coherently in a structural whole. This Paradigm is found manifesting variously in the missionary activities in the forms of proselytisation, Christianisation, modernisation and colonisation. Thus the collections of manuscripts visited by us for the present enquiry becomes part and parcel not only of a cultural event but also of a historical epoch of socio-political dimensions and any study of that age is incomplete without a reference to these collections

Before concluding this report I would like to place on record my deep sense of gratitude to certain individuals and institutions

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CONCLUSION AND ISSUES RAISED

- In the history of mathematics, the invention of calculus and 'the passage to infinity' are seen as major benchmarks in the creation of modern or advanced mathematics. What this volume addresses is the possibility of the beginnings of modern mathematics to have occurred outside Greece and Europe and these ideas and techniques being transmitted to Europe. If such transmission proves to be significant, it would help to deconstruct the prevailing Eurocentric account of the development of modern mathematics.
- In a number of countries, particularly in Asia, there is felt a need to integrate tradition and modernity. The notion of traditional ideas going into the shaping of modernity is important both as a corrective to the conventional explanation of modernity arising solely in Europe as well as offering some measure of 'ownership' in creation of modern science.
- The particular issue of Kerala mathematics raises some difficult methodological issues that needs to be confronted. If transmission is established, the question arises as to whether such a transmission took the form of movement of ideas or of practices or of both.
- If, however, what emerges from research is that parallel development rather than transmission was the case, it becomes no longer viable to explain the genesis of modern mathematics solely in terms of developments in Europe without looking at what was happening in other mathematical traditions.
- After Needham's work on Chinese science, there has been a significant change in the attitude to Chinese civilization. A similar reappraisal of Indian science is long overdue.

- There are different dimensions to the process of transmission. Transmissions have taken place historically in India both within and outside as well during periods before and after the establishment of the British Raj. These need to be examined within the historical context of their occurrence.
- The importance of studying the spread of scientific knowledge should not be minimized. The intellectual assumptions underlying such a study should be scrutinized, particularly if it is accompanied by a form of attribution of origins and retrospective privileging[i] to a specific culture or geographical locality.
- Only one conduit of transmission and one set of data – namely the activities of Jesuits- have been explored in the project. There is need to investigate other possible conduits, including those practices such as cartography, navigation, trade, activities of craftsmen, etc.[ii]
- A whole number of issues are exposed with the opening up of the Pandora's Box of multiple conduits. These include the process of 'underground' transmission through navigators or craftsmen, the importance and presence of the local *ganakas* as mediators between the Brahmins and the populace especially in the construction and use of *panchagam* (astrological calendar), the Arab traders as transmitters, the limitations of a 'brahmin-centric' view of top-down transmissions when there is always the possibility of knowledge at the bottom being transmitted above.....etc. Systems of knowledge should be emphasized not in terms of what is happening to the educated but also how knowledge is used from other groups. Indeed, recent scholarship that has looked at scholar-craftsmen contacts within Europe in the context of the role of experimental philosophy in facilitating the birth of modern science should be expanded to include the possibility of European scholars coming into contact with non-European ideas and practices transmitted to European craftsmen by their non-European counterparts whom they came in contact with through their voyages of discovery in the early modern era.

- Innovations take place both at level of practice as well as the level of theory. Indeed, practice [transmitted through craftsmen] could be as important as theory in the case of movement of ideas and technologies across cultures. Further, identification can structure theory, so that knowledge transmitted through craftsmen across cultures result in the recipient craftsmen developing theories on their own which in turn could inspire others more technically or theoretically qualified to construct their theories. Thus, for example, the transmission of a mathematical practice originating in Kerala through the navigators could in turn be provided their peculiar gloss by the European navigators, who in turn may inspire the European mathematicians to develop their own mathematical explanations when none existed before, which bear little or no resemblance to the original product.
- The project has highlighted the importance of examining in detail the socio-economic context in which Kerala mathematics developed. The important issue is not who did it first, but the historical conditions in which this mathematics arose in a sub-continental context. The lines of communication within the Europe of the period was extensive while the geographical and size constraints in India tended to restrict communication.
- In studying the Kerala context, there is need to examine the role of the *Ganaka* in greater detail, in particular, the social mechanism of parceling out certain activities to the *Ganaka*. There are some interesting questions regarding the social dimensions of ritual and other practices, such as the tendency to leave astrological computations and the construction of the *muhurtas* to the *Ganaka* rather than, say, the Nambuthiri. The Ganaka craftsmen could have transmitted computation techniques rooted in the ideas of Kerala mathematics to European craftsmen which may then have inspired European scholar to make discoveries that paralleled the ideas originated in Kerala.
- There is also a further question on why in Kerala the period between the thirteenth and sixteenth centuries was so productive, resulting a vast amount of literature, compared to other parts of South India, not only in mathematics and

astronomy, but on architecture (*arthashastra*), on *bhakti* traditions and medicine (*ayurveda*) apart from literary compositions. Could this be explained by increase in state patronage? Did the development of Malayalam as a written language during this period have any bearing on the emergence of this 'golden period' in Kerala?

- Studies of the Jesuits as scientific mediators/missionaries in India on the basis of Jesuit archives in India has been sparse, consisting of no more than twenty items. The actions of Marquess de Pomba, in 1759, of confiscating Jesuit possessions and destroying Jesuit records have not helped. More studies have been made regarding activities in China with the Jesuits there being in charge of the Astronomical Bureau and acting as scientific advisers for about two centuries. As far as India, what remains are mainly astronomical observations made during the 18th century and general descriptions of Hindustan. None of the 'missionary' archives either in India or outside contain scientific information. There are also papers in the Jesuit Archives at Kodaikanal to indicate the close relationship between members of the Court of Cochin and the Portuguese.

ENDNOTES

1 'Retrospective Privileging' implies 'looking back' and granting a special place or privilege to a certain group or culture The most powerful and sustained form of retrospective privileging has been the Eurocentric Vision, the centre-piece of which are the existence of a 'Greek Miracle' and the appearance of the 'Dark Ages'.

2 There were of course as well as ideas-mediated transmission through Arab and Chinese intermediaries to Europeans, which need to considered.

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